Direct Method of Interpolation

Computer Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates
Direct Method of Interpolation

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What is Interpolation?

Given \((x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\), find the value of ‘y’ at a value of ‘x’ that is not given.

**Figure 1** Interpolation of discrete.
Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate
Direct Method

Given ‘n+1’ data points \((x_0,y_0), (x_1,y_1), \ldots, (x_n,y_n)\), pass a polynomial of order ‘n’ through the data as given below:

\[
y = a_0 + a_1 x + \ldots + a_n x^n.
\]

where \(a_0, a_1, \ldots, a_n\) are real constants.

- Set up ‘n+1’ equations to find ‘n+1’ constants.
- To find the value ‘y’ at a given value of ‘x’, simply substitute the value of ‘x’ in the above polynomial.
Example

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a rectangular plate. The hole centers in the plate that describe the path the arm needs to take are given below.

If the laser is traversing from $x = 2$ to $x = 4.25$ in a linear path, find the value of $y$ at $x = 4$ using the direct method for linear interpolation.

<table>
<thead>
<tr>
<th>$x$ (m)</th>
<th>$y$ (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>7.2</td>
</tr>
<tr>
<td>4.25</td>
<td>7.1</td>
</tr>
<tr>
<td>5.25</td>
<td>6.0</td>
</tr>
<tr>
<td>7.81</td>
<td>5.0</td>
</tr>
<tr>
<td>9.2</td>
<td>3.5</td>
</tr>
<tr>
<td>10.6</td>
<td>5.0</td>
</tr>
</tbody>
</table>

Figure 2 Location of holes on the rectangular plate.
Linear Interpolation

\[ y(x) = a_0 + a_1 x \]
\[ y(2.00) = a_0 + a_1 (2.00) = 7.2 \]
\[ y(4.25) = a_0 + a_1 (4.25) = 7.1 \]

Solving the above two equations gives,

\[ a_0 = 7.2889 \quad a_1 = -0.044444 \]

Hence

\[ y(x) = 7.2889 - 0.044444x, \quad 2.00 \leq x \leq 4.25 \]
\[ y(4.00) = 7.2889 - 0.044444(4.00) = 7.11111 \text{ in.} \]
Example

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a rectangular plate. The hole centers in the plate that describe the path the arm needs to take are given below.

If the laser is traversing from $x = 2$ to $x = 4.25$ in a linear path, find the value of $y$ at $x = 4$ using the direct method for quadratic interpolation.

<table>
<thead>
<tr>
<th>$x$ (m)</th>
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</tr>
</thead>
<tbody>
<tr>
<td>2</td>
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</tr>
<tr>
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</tr>
</tbody>
</table>

Figure 2 Location of holes on the rectangular plate.
Quadratic Interpolation

\[ y(x) = a_0 + a_1 x + a_2 x^2 \]

\[ y(2.00) = a_0 + a_1 (2.00) + a_2 (2.00)^2 = 7.2 \]
\[ y(4.25) = a_0 + a_1 (4.25) + a_2 (4.25)^2 = 7.1 \]
\[ y(5.25) = a_0 + a_1 (5.25) + a_2 (5.25)^2 = 6.0 \]

Solving the above three equations gives

\[ a_0 = 4.5282 \quad a_1 = 1.9855 \quad a_2 = -0.32479 \]
Quadratic Interpolation (contd)

\[ y(x) = 4.5282 + 1.9855x - 0.32479x^2, \quad 2.00 \leq x \leq 5.25 \]
\[ y(4.00) = 4.5282 + 1.9855(4.00) - 0.32479(4.00)^2 \]

\[ = 7.2735 \text{ in.} \]

The absolute relative approximate error \( |\epsilon_a| \) obtained between first and second order polynomial is

\[ |\epsilon_a| = \left| \frac{7.2735 - 7.1111}{7.2735} \right| \times 100 \]

\[ = 2.2327\% \]
## Comparison Table

<table>
<thead>
<tr>
<th>Order of Polynomial</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Location (in.)</td>
<td>7.1111</td>
<td>7.2735</td>
</tr>
<tr>
<td>Absolute Relative Approximate Error</td>
<td>-------</td>
<td>2.2327%</td>
</tr>
</tbody>
</table>
Example

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a rectangular plate. The hole centers in the plate that describe the path the arm needs to take are given below.

If the laser is traversing from $x = 2$ to $x = 4.25$ in a linear path, find the value of $y$ at $x = 4$ using the direct method using a fifth order polynomial.

<table>
<thead>
<tr>
<th>$x$ (m)</th>
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<tbody>
<tr>
<td>2</td>
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</table>
Fifth Order Interpolation

\[ y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 \]

\[
y(2.00) = 7.2 = a_0 + a_1 (2.00) + a_2 (2.00)^2 + a_3 (2.00)^3 + a_4 (2.00)^4 + a_5 (2.00)^5
\]

\[
y(4.25) = 7.1 = a_0 + a_1 (4.25) + a_2 (4.25)^2 + a_3 (4.25)^3 + a_4 (4.25)^4 + a_5 (4.25)^5
\]

\[
y(5.25) = 6.0 = a_0 + a_1 (5.25) + a_2 (5.25)^2 + a_3 (5.25)^3 + a_4 (5.25)^4 + a_5 (5.25)^5
\]

\[
y(7.81) = 5.0 = a_0 + a_1 (7.81) + a_2 (7.81)^2 + a_3 (7.81)^3 + a_4 (7.81)^4 + a_5 (7.81)^5
\]

\[
y(9.20) = 3.5 = a_0 + a_1 (9.20) + a_2 (9.20)^2 + a_3 (9.20)^3 + a_4 (9.20)^4 + a_5 (9.20)^5
\]

\[
y(10.60) = 5.0 = a_0 + a_1 (10.60) + a_2 (10.60)^2 + a_3 (10.60)^3 + a_4 (10.60)^4 + a_5 (10.60)^5
\]
Fifth Order Interpolation (contd)

Writing the six equations in matrix form, we have

\[
\begin{bmatrix}
1 & 2.00 & 4.00 & 8.00 & 16.00 & 32 \\
1 & 4.25 & 18.063 & 76.766 & 326.25 & 1386.6 \\
1 & 5.25 & 27.563 & 144.70 & 759.69 & 3988.4 \\
1 & 7.81 & 60.996 & 476.38 & 3720.5 & 29057 \\
1 & 9.20 & 84.640 & 778.69 & 7163.9 & 65908 \\
1 & 10.6 & 112.36 & 1191.0 & 12625 & 133820 \\
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5 \\
\end{bmatrix}
= \begin{bmatrix} 7.2 \\
7.1 \\
6.0 \\
5.0 \\
3.5 \\
5.0 \end{bmatrix}
\]

\[a_0 = -30.898 \quad a_1 = 41.344 \quad a_2 = -15.855\]

\[a_3 = 2.7862 \quad a_4 = -0.23091 \quad a_5 = 0.0072923\]

\[y(x) = -30.898 + 41.344x - 15.855x^2 + 2.7862x^3 - 0.23091x^4 + 0.0072923x^5, 2 \leq x \leq 10.6\]
Fifth Order Interpolation (contd)

\[ y(x) = -30.898 + 41.344x - 15.855x^2 + 2.7862x^3 - 0.23091x^4 + 0.0072923x^5, \quad 2 \leq x \leq 10.6 \]
Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/direct_method.html
THE END

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