Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
Shooting Method
Ordinary Differential Equations
COMPLETE SOLUTION SET

1. The exact solution to the boundary value problem

$$\frac{d^2y}{dx^2} = 6x - 0.5x^2,\ y(0) = 0,\ y(12) = 0$$

for $y(4)$ is

(A) -234.66
(B) 0.00
(C) 16.000
(D) 106.66

Solution

The correct answer is (A)

First separate the variables

$$\frac{d^2y}{dx^2} = 6x - 0.5x^2$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 6x - 0.5x^2$$

$$\frac{dy}{dx} = \int \left(6x - 0.5x^2\right)dx$$

$$= \frac{6x^2}{2} - \frac{0.5x^2}{3} + C$$

$$= 3x^2 - \frac{0.5}{3}x^3 + C$$

$$y = \int \left(3x^2 - \frac{0.5}{3}x^3 + C\right)dx$$

$$= \frac{3}{3}x^3 - \frac{0.5}{12}x^4 + Cx + D$$

$$= x^3 - \frac{0.5}{12}x^4 + Cx + D$$

Set $y(0)=0$, to solve for D

$$y(0)=0^3-\frac{0.5}{12}\times 0^4+C\times 0+D$$

$$0=D$$

Set $y(12)=0$, to solve for C

$$y(12)=12^3-\frac{0.5}{12}\times 12^4+C\times 12$$

$$0=1728-864+12C$$

$$-12C=864$$

$$C=-72$$

Thus

$$y=x^3-\frac{0.5}{12}x^4+Cx+D$$

$$=x^3-\frac{0.5}{12}x^4-72x$$

$$y(4)=4^3-\frac{0.5}{12}\times 4^4-72\times 4$$

$$=64-10.6667-288$$

$$=-234.667$$

2. Given

$$\frac{d^2y}{dx^2} = 6x - 0.5x^2,\ y(0) = 0,\ y(12) = 0, \text{ the exact value of } \frac{dy}{dx}(0) \text{ is}$$

(A) -72.0
(B) 0.00
(C) 36.0
(D) 72.0

Solution

The correct answer is (A)

First separate the variables

$$\frac{d^2y}{dx^2} = 6x - 0.5x^2$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 6x - 0.5x^2$$

$$\frac{dy}{dx} = \int \left(6x - 0.5x^2\right)dx$$

$$= \frac{6x^2}{2} - \frac{0.5x^2}{3} + C$$

$$= 3x^2 - \frac{0.5}{3}x^3 + C$$

$$y = \int \left(3x^2 - \frac{0.5}{3}x^3 + C\right)dx$$

$$= \frac{3}{3}x^3 - \frac{0.5}{12}x^4 + Cx + D$$

$$= x^3 - \frac{0.5}{12}x^4 + Cx + D$$

Set $y(0) = 0$, to solve for D

$$y(0) = 0^3 - \frac{0.5}{12} \times 0^4 + C \times 0 + D$$

$$0 = D$$

Set $y(12) = 0$, to solve for C

$$y(12) = 12^3 - \frac{0.5}{12} \times 12^4 + C \times 12$$

$$0 = 1728 - 864 + 12C$$

$$-12C = 864$$

$$C = -72$$

Then take the derivative of both sides

$$y = x^3 - \frac{0.5}{12}x^4 + Cx + D$$

$$= x^3 - \frac{0.5}{12}x^4 - 72x$$

$$\frac{dy}{dx} = 3x^2 - \frac{0.5}{3}x^3 - 72$$

$$\frac{dy}{dx}(0) = 3(0)^2 - \frac{0.5}{3}(0)^3 - 72$$

$$= -72$$

3. Given

$$\frac{d^2y}{dx^2} = 6x - 0.5x^2, \; y(0) = 0, \; y(12) = 0,$$

If one was using shooting method with Euler's method with a step size of $h = 4$, and an assumed value of $\frac{dy}{dx}(0)$=20, then the estimated value of $y(12)$ in the first iteration most nearly is

(A) 60.0
(B) 496.0
(C) 1088
(D) 1102

Solution

The correct answer is (B)

$$\frac{d^2y}{dx^2} = 6x - 0.5x^2$$

Let

$$\frac{dy}{dx} = z$$

$$\frac{dz}{dx} = 6x - 0.5x^2$$

$$\frac{dy}{dx} = z = f_1(x, y, z), y(0) = 0$$

$$\frac{dz}{dx} = 6x - 0.5x^2 = f_2(x, y, z), \frac{dy}{dx}(0) = z(0) = 20$$

$$y_{i+1} = y_i + f_1(x_i, y_i, z_i)h$$

$$z_{i+1} = z_i + f_2(x_i, y_i, z_i)h$$

For $i = 0, x_0 = 0, y_0 = 0, z_0 = 20$

$$x_1 = x_0 + h$$
$$= 0 + 4$$
$$= 4$$

$$y_1 = y_0 + f(x_0, y_0, z_0)h$$
$$= 0 + f(0,0,20) \times 4$$
$$= 0 + (20) \times 4$$
$$= 80$$

$$z_1 = z_0 + f_2(x_0, y_0, z_0)h$$
$$= 20 + f_2(0,0,20) \times 4$$
$$= 20$$

For $i = 1, x_1 = 4, y_1 = 80, z_1 = 20$

$$
\begin{aligned}
x_2 &= x_1 + h \\
&= 4 + 4 \\
&= 8 \\
y_2 &= y_1 + f(x_1, y_1, z_1)h \\
&= 80 + f(4,80,20) \times 4 \\
&= 80 + (20) \times 4 \\
&= 160 \\
z_2 &= z_1 + f_2(x_1, y_1, z_1)h \\
&= 20 + f_2(4,8,20) \times 4 \\
&= 20 + \left(6 \times 4 - 0.5 \times 4^2\right) \times 4 \\
&= 84
\end{aligned}
$$

For $i = 2, x_2 = 8, y_2 = 160, z_2 = 84$

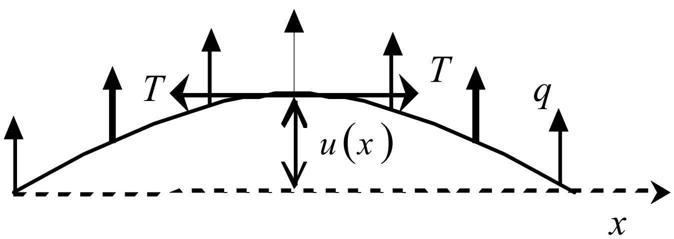
$$
\begin{aligned}
x_3 &= x_2 + h \\
&= 8 + 4 \\
&= 12 \\
y_3 &= y_2 + f(x_2, y_2, z_2)h \\
&= 160 + f(8,160,84) \times 4 \\
&= 160 + (84) \times 4 \\
&= 496 \\
z_3 &= z_2 + f_2(x_2, y_2, z_2)h \\
&= 84 + f_2(8,160,84) \times 4 \\
&= 84 + \left(6 \times 8 - 0.5(8)^2\right) \times 4 \\
&= 148 \\
y_3 &\cong y(x_3) \\
&= y(12) \\
&= 496
\end{aligned}
$$

4. The transverse deflection, u of a cable of length, L, fixed at both ends, is given as a solution to

$$\frac{d^2u}{dx^2} = \frac{Tu}{R} + \frac{qx(x-L)}{2R}$$

where

T = tension in cable
R = flexural stiffness
q = distributed transverse load



Given are $L = 50"$, $T = 2000 lbs$, $q = 75\frac{lbs}{in}$, $R = 75\times10^6 lbs\cdot in^2$. The shooting method is used with Euler's method assuming a step size of $h = 12.5"$. Initial slope guesses at $x = 0$ of $\frac{du}{dx} = 0.003$ and $\frac{du}{dx} = 0.004$ are used in order, and then refined for the next iteration using linear interpolation after the value of $u(L)$ is found. The deflection in inches at the center of the cable found during the second iteration is most nearly

(A) 0.075000
(B) 0.10000
(C) -0.061291
(D) 0.00048828

Solution

The correct answer is (C)

$$\frac{d^2u}{dx^2} = \frac{Tu}{R} + \frac{qx(x-L)}{2R}$$

Let

$$\frac{du}{dx} = z = f_1(x,u,z)$$

$$\frac{dz}{dx} = \frac{Tu}{R} + \frac{q\times x(x-L)}{2R}$$

$$= \frac{2000u}{75\times10^6} + \frac{75\times x(x-L)}{2(75\times10^6)}$$

$$= f_1(x,u,z)$$

$$u_{i+1} = u_i + f_1(x_i,u_i,z_i)h$$

$$z_{i+1} = z_i + f_2(x_i,u_i,z_i)h$$

The first iteration

For $i=0, x_0=0, u_0=0, z_0=0.003$

$$
\begin{aligned}
u_1 &= u_0 + f_1(x_0, u_0, z_0)12.5 \\
&= 0 + f_1(0,0,0.003)\times 12.5 \\
&= 0 + (0.003)\times 12.5 \\
&= 0.0375
\end{aligned}
$$

$$
\begin{aligned}
z_1 &= z_0 + f_2(x_0, u_0, z_0)h \\
&= 0.003 + f_2(0,0,0.003)\times 12.5 \\
&= 0.003
\end{aligned}
$$

For $i=1, x_1=12.5, u_1=0.0375, z_1=0.003$

$$
\begin{aligned}
u_2 &= u_1 + f_1(x_1, u_1, z_1)12.5 \\
&= 0.0375 + f_1(12.5,0.0375,0.003)\times 12.5 \\
&= 0.0375 + (0.003)\times 12.5 \\
&= 0.075
\end{aligned}
$$

$$
\begin{aligned}
z_2 &= z_1 + f_2(x_1, u_1, z_1)h \\
&= 0.003 + f_2(12.5,0.0375,0.003)\times 12.5 \\
&= 0.003 + \left(\frac{-2000\times 0.0375}{75\times 10^6} - \frac{75\times 12.5(12.5-50)}{2(75\times 10^6)}\right)\times 12.5 \\
&= 0.005917
\end{aligned}
$$

For $i=2, x_2=25, u_2=0.075, z_2=0.005917$

$$
\begin{aligned}
u_3 &= u_2 + f_1(x_2, u_2, z_2)12.5 \\
&= 0.075 + f_1(25,0.075,0.005917)\times 12.5 \\
&= 0.075 + (0.005917)\times 12.5 \\
&= 0.14896
\end{aligned}
$$

$$
\begin{aligned}
z_3 &= z_2 + f_2(x_2, u_2, z_2)h \\
&= 0.005917 + f_2(25,0.075,7\times 10^{-5})\times 12.5 \\
&= 0.005917 + \left(\frac{-2000\times 0.075}{75\times 10^6} - \frac{75\times 25(25-50)}{2(75\times 10^6)}\right)\times 12.5 \\
&= 0.0097984
\end{aligned}
$$

For $i=3, x_3=37.5, u_3=0.14896, z_3=0.0097984$

$$
\begin{aligned}
u_4 &= u_3 + f_1(x_3, u_3, z_3)12.5 \\
&= 0.14896 + f_1(37.5,0.14896,0.0097984)\times 12.5 \\
&= 0.14896 + (0.0097984)\times 12.5 \\
&= 0.27145
\end{aligned}
$$

$$z_4 = z_3 + f_2(x_3, u_3, z_3)h$$
$$= 0.0097984 + f_2(37.5, 0.14896, 0.0097984) \times 12.5$$
$$= 0.0097984 + \left(\frac{-2000 \times 014896}{75 \times 10^6} - \frac{75 \times 37.5(37.5 - 50)}{2(75 \times 10^6)} \right) \times 12.5$$
$$= 0.012678$$

The second iteration

For $i = 0, x_0 = 0, u_0 = 0, z_0 = 0.004$

$$u_1 = u_0 + f_1(x_0, u_0, z_0)12.5$$
$$= 0 + f_1(0,0,0.004) \times 12.5$$
$$= 0 + (0.004) \times 12.5$$
$$= 0.05$$
$$z_1 = z_0 + f_2(x_0, u_0, z_0)h$$
$$= 0.004 + f_2(0,0,0.004) \times 12.5$$
$$= 0.004$$

For $i = 1, x_1 = 12.5, u_1 = 0.05, z_1 = 0.004$

$$u_2 = u_1 + f_1(x_1, u_1, z_1)12.5$$
$$= 0.05 + f_1(12.5, 0.05, 0.004) \times 12.5$$
$$= 0.05 + (0.004) \times 12.5$$
$$= 0.1$$
$$z_2 = z_1 + f_2(x_1, u_1, z_1)h$$
$$= 0.004 + f_2(12.5, 0.05, 0.004) \times 12.5$$
$$= 0.004 + \left(\frac{-2000 \times 0.05}{75 \times 10^6} - \frac{75 \times 12.5(12.5 - 50)}{2(75 \times 10^6)} \right) \times 12.5$$
$$= 0.006913$$

For $i = 2, x_2 = 25, u_2 = 0.1, z_2 = 0.006913$

$$u_3 = u_2 + f_1(x_2, u_2, z_2)12.5$$
$$= 0.1 + f_1(25, 0.1, 0.006913) \times 12.5$$
$$= 0.1 + 0.006913 \times 12.5$$
$$= 0.18641$$
$$z_3 = z_2 + f_2(x_2, u_2, z_2)h$$
$$= 0.006913 + f_2(25, 0.1, 0.006913) \times 12.5$$
$$= 0.006913 + \left(\frac{-2000 \times 0.1}{75 \times 10^6} - \frac{75 \times 25(25 - 50)}{2(75 \times 10^6)} \right) \times 12.5$$
$$= 0.010786$$

For $i = 3, x_3 = 37.5, u_3 = 0.18641, z_3 = 0.010786$

$$
\begin{aligned}
u_4 &= u_3 + f_1(x_3, u_3, z_3)12.5 \\
&= 0.18641 + f_1(37.5, 0.18641, 0.010786) \times 12.5 \\
&= 0.18641 + (0.010786) \times 12.5 \\
&= 0.32124
\end{aligned}
$$

$$
\begin{aligned}
z_4 &= z_3 + f_2(x_3, u_3, z_3)h \\
&= 0.010786 + f_2(37.5, 0.18641, 0.010786) \times 12.5 \\
&= 0.010786 + \left(\frac{2000 \times 0.18641}{75 \times 10^6} + \frac{75 \times 37.5(37.5 - 50)}{2(75 \times 10^6)} \right) \times 12.5 \\
&= 0.013653
\end{aligned}
$$

linear interpolation

$$
\frac{du}{dr}(50) \cong \frac{0.003 - 0.004}{0.271445 - 0.32124}(0 - 0.32124) + (0.004)
$$

$$
\frac{du}{dr}(50) \cong -0.00245162
$$

For $i = 0, x_0 = 0, u_0 = 0, z_0 = \text{-}0.0024516$

$$
\begin{aligned}
u_1 &= u_0 + f_1(x_0, u_0, z_0)12.5 \\
&= 0 + f_1(0, 0, -0.0024516) \times 12.5 \\
&= 0 + (-0.0024516) \times 12.5 \\
&= -0.030645
\end{aligned}
$$

$$
\begin{aligned}
z_1 &= z_0 + f_2(x_0, u_0, z_0)h \\
&= -0.0024516 + f_2(0, 0, 0.0024516) \times 12.5 \\
&= -0.0024516
\end{aligned}
$$

For $i = 1, x_1 = 12.5, u_1 = -0.030645, z_1 = \text{-}\ 0.0024516$

$$
\begin{aligned}
u_2 &= u_1 + f_1(x_1, u_1, z_1)12.5 \\
&= -0.030645 + f_1(12.5, -0.030645, -0.0024516) \times 12.5 \\
&= -0.030645 + (-0.0024516) \times 12.5 \\
&= -0.061291
\end{aligned}
$$

$$
\begin{aligned}
z_2 &= z_1 + f_2(x_1, u_1, z_1)h \\
&= -0.0024516 + f_2(12.5, -0.030645, -0.0024516) \times 12.5 \\
&= -0.0024516 + \left(\frac{-2000 \times 0.030645}{75 \times 10^6} - \frac{75 \times 12.5(12.5 - 50)}{2(75 \times 10^6)} \right) \times 12.5 \\
&= 0.00048828
\end{aligned}
$$

For $i = 2, x_2 = 25, u_2 = -0.061291, z_2 = 0.00048828$

$$
\begin{aligned}
u_3 &= u_2 + f_1(x_2, u_2, z_2)12.5 \\
&= -0.061291 + f_1(25, 0.061291, 0.00048828) \times 12.5 \\
&= -0.061291 + (0.00048828) \times 12.5 \\
&= -0.055187
\end{aligned}
$$

$$
\begin{aligned}
z_3 &= z_2 + f_2(x_2, u_2, z_2)h \\
&= 0.00048828 + f_2(25, -0.061291, 0.00048828) \times 12.5 \\
&= 0.00048828 + \left(\frac{-2000 \times -0.06105}{75 \times 10^6} - \frac{75 \times 25(25-50)}{2(75 \times 10^6)} \right) \times 12.5 \\
&= 0.004415
\end{aligned}
$$

For $i = 3, x_3 = 37.5, u_3 = -0.055187, z_3 = 0.004415$

$$
\begin{aligned}
u_4 &= u_3 + f_1(x_3, u_3, z_3)12.5 \\
&= -0.055187 + f_1(37.5, 0.055187, 0.004415) \times 12.5 \\
&= -0.055187 + (0.004415) \times 12.5 \\
&= -2.4286 \times 10^{-16}
\end{aligned}
$$

thus the value of u at $x = 25$

$$u(25) = -0.061291''$$

Now we can use this new approximation for the derivative $\frac{dy}{dx}$ to solve another Euler problem.
Comparing the result with the boundary condition, we find

$$
\begin{aligned}
u_4 &= -2.4286 \times 10^{-16}{}'' \\
u(50) &= 0'' \\
E_t &= u_4 - u(50) \\
&= -2.4286 \times 10^{-16}
\end{aligned}
$$

5. The radial displacement, u is a pressurized hollow thick cylinder (inner radius=5″, outer radius=8″) is given at different radial locations.

Radius	**Radial Displacement**
(in)	**(in)**
5.0	0.0038731
5.6	0.0036165
6.2	0.0034222
6.8	0.0032743
7.4	0.0031618
8.0	0.0030769

The maximum normal stress, in psi, on the cylinder is given by

$$\sigma_{\max} = 3.2967 \times 10^{6}\left(\frac{u(5)}{5} + 0.3\frac{du}{dr}(5)\right)$$

The maximum stress, in psi, with second order accuracy is

(A) 2079.3
(B) 2104.5
(C) 2130.7
(D) 2182.0

Solution

The correct answer is (A)

If we look at the Talyor series

$$u(r+\Delta r) = u(r) + u'(r)\Delta r + \frac{u''(r)(\Delta r)^2}{2} + \frac{u'''(r)(\Delta r)^3}{6} + \cdots, \text{ and}$$

$$u(r+2\Delta r) = u(r) + u'(r)(2\Delta r) + \frac{u''(r)2\Delta r^2}{2} + \frac{u'''(r)(2\Delta r)^3}{6} + \cdots$$

Multiply the first by 4 and subtract it from the second

$$u(r+2\Delta r) - 4u(r+\Delta r) = -3u(r) + u'(r)2\Delta r + 0(\Delta r)^3$$

$$u'(r)2\Delta r = -u(r+2\Delta r) + 4u(r+\Delta r) - 3u(r) + 0(\Delta r)^3$$

$$u'(r) = \frac{-u(r+2\Delta r) + 4u(r+\Delta r) - 3u(r)}{2\Delta r} + \frac{0(\Delta r)^3}{2\Delta r}$$

$$u'(r) = \frac{-u(r+2\Delta r) + 4u(r+\Delta r) - 3u(r)}{2\Delta r} + 0(\Delta r)^2$$

This equation is second order accurate, that is , the true error is $O\left((\Delta r)^2\right)$

$$\frac{du}{dr} = \frac{-u(r+2\Delta r) + 4u(r+\Delta r) - 3u(r)}{2\Delta r}$$

$$r = 5, \Delta r = 0.6$$

$$\frac{du}{dr}(5) = \frac{-u(5\{+2\times 0.6) + 4u(5+0.6) - 3u(5)}{2\times 0.6}$$

$$\frac{du}{dr}(5) = \frac{-u(6.2) + 4u(5.6) - 3u(5)}{2\times 0.6}$$

$$= \frac{-0.0034222 + 0.014466 - 0.011619}{1.2}$$

$$= -0.00047933$$

Thus,

$$\sigma_{\max} = 3.2967\times 10^{6}\left(\frac{u(5)}{5} + 0.3\frac{du}{dr}(5)\right)$$

$$= 3.2967\times 10^{6}\left(\frac{0.0038731}{5} + 0.3\times(-0.00047933)\right)$$

$$= 2079.6 \ psi$$

6. For a simply supported beam (at $x = 0$ and $x = L$) with a uniform load q, the vertical deflection $v(x)$ is described by the boundary value ordinary differential equation as

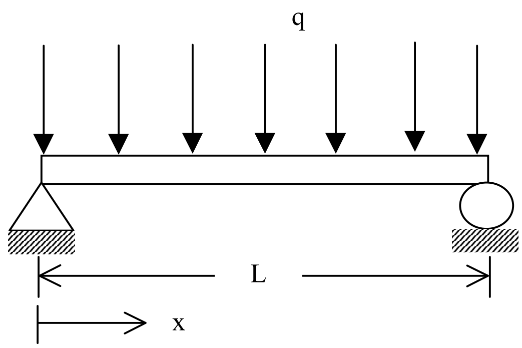
$$\frac{d^2v}{dx^2} = \frac{qx(x-L)}{2EI},\ 0 \le x \le L$$

where

E = Young's modulus of elasticity of beam

I = second moment of area.

This ordinary differential equations is based on assuming that $\frac{dv}{dx}$ is small. If $\frac{dv}{dx}$ is not small, then the ordinary differential equation is



(A) $$\frac{\frac{d^2v}{dx^2}}{\sqrt{1+\left(\frac{dv}{dx}\right)^2}} = \frac{qx(x-L)}{2EI}$$

(B) $$\frac{\frac{d^2v}{dx^2}}{\left(1+\left(\frac{dv}{dx}\right)^2\right)^{3/2}} = \frac{qx(x-L)}{2EI}$$

(C) $$\frac{\frac{d^2v}{dx^2}}{\sqrt{1+\left(\frac{dv}{dx}\right)}} = \frac{qx(x-L)}{2EI}$$

(D) $$\frac{\frac{d^2v}{dx^2}}{1+\frac{dv}{dx}} = \frac{qx(x-L)}{2EI}$$

Solution

The correct answer is (B)

The equation for the deflection in a beam is

$$\frac{1}{\rho} = \frac{M}{EI}$$

where

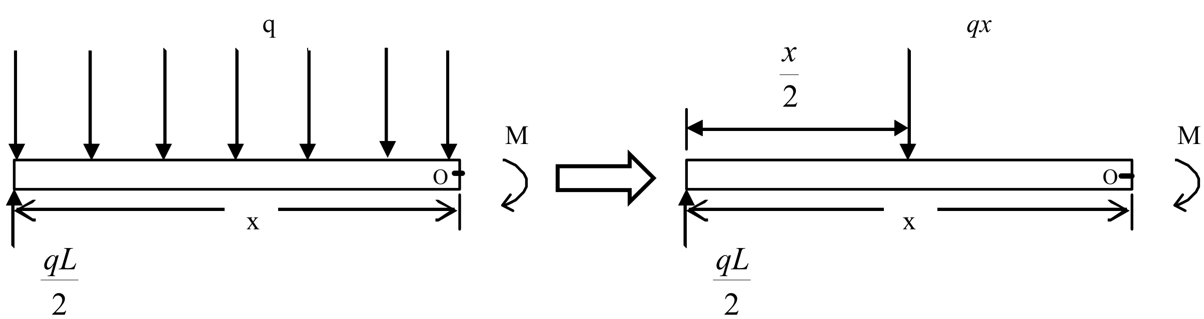
$$\frac{1}{\rho} = \text{ the curvature}$$

M = the internal bending moment of the beam where the curvature is to be determined

The curvature can be rewritten in rectangular format as

$$\frac{1}{\rho} = \frac{\frac{d^2v}{dx^2}}{\left(1+\left(\frac{dv}{dx}\right)^2\right)^{3/2}}$$

The reaction at each support is $\frac{qL}{2}$.



writing the balance of bending moments at point O at a distance x from the midpoint

$$M + \frac{qL}{2}x - (qx)\frac{x}{2} = 0$$

$$M = \frac{qx^2}{2} - \frac{qL}{2}x$$

$$= \frac{qx(x-L)}{2}$$

Thus the equation for the deflection of a simple supported beam is

$$\frac{\frac{d^2v}{dx^2}}{\left(1+\left(\frac{dv}{dx}\right)^2\right)^{3/2}} = \frac{qx(x-L)}{2EI}$$