Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
Runge-Kutta 2nd Order Method
Ordinary Differential Equations
COMPLETE SOLUTION SET

1. To solve the ordinary differential equation

$$3\frac{dy}{dx}+xy^2=\sin x,\ \ y(0)=5$$

by the Runge-Kutta 2nd order method, you need to rewrite the equation as

(A) $\frac{dy}{dx}=\sin x-xy^2,\ \ y(0)=5$

(B) $\frac{dy}{dx}=\frac{1}{3}\left(\sin x-xy^2\right),\ \ y(0)=5$

(C) $\frac{dy}{dx}=\frac{1}{3}\left(-\cos x-\frac{xy^3}{3}\right),\ \ y(0)=5$

(D) $\frac{dy}{dx}=\frac{1}{3}\sin x,\ \ y(0)=5$

Solution

The correct answer is (B).

To solve ordinary differential equations by the Runge-Kutta 2nd order method, you need to rewrite the equation in the following form

$$\frac{dy}{dx}=f(x,y),\ \ y(0)=y_0$$

Thus,

$$3\frac{dy}{dx}+xy^2=\sin x,\ \ y(0)=5$$

$$3\frac{dy}{dx}=\sin x-xy^2,\ \ y(0)=5$$

$$\frac{dy}{dx}=\frac{1}{3}\left(\sin x-xy^2\right),\ \ y(0)=5$$

2. Given

$$3\frac{dy}{dx}+5y^2=\sin x,\ \ y(0.3)=5$$

and using a step size of $h=0.3$, the value of $y(0.9)$ using the Runge-Kutta 2nd order Heun method is most nearly

(A) -4297.4
(B) -4936.7
(C) -0.21336×10^{14}
(D) -0.24489×10^{14}

Solution

The correct answer (A).

$$3\frac{dy}{dx}+5y^2=\sin x,\ \ y(0.3)=5$$

is rewritten as

$$\frac{dy}{dx}=\frac{1}{3}\left(\sin x-5y^2\right)=f(x,y)$$

$$f(x,y)=\frac{1}{3}\left(\sin x-5y^2\right)$$

In Huen's method $a_2=\frac{1}{2}$ is chosen, giving

$$a_1=\frac{1}{2}$$

$$p_1=1$$

$$q_{11}=1$$

resulting in

$$y_{i+1}=y_i+\left(\frac{1}{2}k_1+\frac{1}{2}k_2\right)h$$

where

$$k_1=f(x_i,y_i)$$

$$k_2=f(x_i+h,y_i+k_1h)$$

$$h=0.3$$

for $i=0,\ x_0=0.3,\ \ y_0=5$

$$\begin{aligned}k_1&=f(x_0,y_0)\\&=f(0.3,5)\\&=\frac{1}{3}\left(\sin(0.3)-5(5)^2\right)\\&=-41.5682\end{aligned}$$

$$k_2 = f(x_0 + h, y_0 + k_1 h)$$
$$= f(0.3 + 0.3, 5 + (-41.5682 \times 0.3))$$
$$= f(0.6, -7.4704)$$
$$= \frac{1}{3}\left(\sin(0.6) - 5(-7.4704)^2\right)$$
$$= \frac{1}{3}(0.56464 - 279.04)$$
$$= -92.824$$
$$y_1 = y_0 + \left(\frac{1}{2}(k_1) + \frac{1}{2}(k_2)\right)h$$
$$= 5 + \left(\frac{1}{2}(-41.5682) + \frac{1}{2}(-92.824)\right)0.3$$
$$= 5 + (-67.196) \times 0.3$$
$$= -15.159$$
$$x_1 = x_0 + h$$
$$= 0.3 + 0.3$$
$$= 0.6$$
$$y(0.6) \approx y_1 = -15.159$$

for $i = 1$, $x_1 = 0.6$, $y_1 = -15.159$

$$k_1 = f(x_1, y_1)$$
$$= f(0.6, -15.159)$$
$$= \frac{1}{3}\left(\sin(0.6) - 5(-15.159)^2\right)$$
$$= -382.80$$
$$k_2 = f(x_1 + h, y_1 + k_1 h)$$
$$= f(0.6 + 0.3, -15.159 + (-382.80 \times 0.3))$$
$$= f(0.9, -130.00)$$
$$= \frac{1}{3}\left(\sin(0.9) - 5(-130.00)^2\right)$$
$$= \frac{1}{3}(0.78333 - 84500)$$
$$= -28166$$

$$
\begin{aligned}
y_2 &= y_1 + \left(\frac{1}{2}(k_1) + \frac{1}{2}(k_2)\right)h \\
&= -15.159 + \left(\frac{1}{2}(-382.80) + \frac{1}{2}(-28166)\right)0.3 \\
&= -15.159 + (-14274) \times 0.3 \\
&= -4297.4
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= x_1 + h \\
&= 0.6 + 0.3 \\
&= 0.9
\end{aligned}
$$

$$
y(0.9) \approx y_2 = -4297.4
$$

3. Given

$$3\frac{dy}{dx}+5\sqrt{y}=e^{0.1x},\ \ y(0.3)=5$$

and using a step size of $h=0.3$, the best estimate of $\frac{dy}{dx}(0.9)$ using the Runge-Kutta 2nd order midpoint method most nearly is

(A) –2.2473
(B) –2.2543
(C) –2.6188
(D) –3.2045

Solution

The correct answer is (C).

$$3\frac{dy}{dx}+5\sqrt{y}=e^{0.1x}$$

is rewritten as

$$\frac{dy}{dx}=\frac{1}{3}\left(e^{0.1x}-5\sqrt{y}\right)=f(x,y)$$

$$f(x,y)=\frac{1}{3}\left(e^{0.1x}-5\sqrt{y}\right)$$

In the midpoint method $a_2=1$ is chosen, giving

$$a_1=0$$

$$p_1=\frac{1}{2}$$

$$q_{11}=\frac{1}{2}$$

resulting in

$$y_{i+1}=y_i+k_2h$$

where

$$k_1=f(x_i,y_i)$$

$$k_2=f\left(x_i+\frac{1}{2}h,y_i+\frac{1}{2}k_1h\right)$$

$$h=0.3$$

for $i=0,\ x_0=0.3,\ y_0=5$

$$k_1 = f(x_0, y_0)$$
$$= f(0.3, 5)$$
$$= \frac{1}{3}\left(e^{0.1\times0.3} - 5\sqrt{5}\right)$$
$$= \frac{1}{3}(1.0305 - 11.180)$$
$$= -3.3833$$
$$k_2 = f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1 h\right)$$
$$= f\left(0.3 + \frac{0.3}{2}, 5 + \frac{1}{2}(-3.3833\times0.3)\right)$$
$$= f(0.45, 4.4925)$$
$$= \frac{1}{3}\left(e^{0.1\times0.45} - 5\sqrt{4.4925}\right)$$
$$= \frac{1}{3}(1.0460 - 10.598)$$
$$= -3.1839$$
$$y_1 = y_0 + k_2 h$$
$$= 5 + (-3.1839)\times 0.3$$
$$= 4.0448$$
$$x_1 = x_0 + h$$
$$= 0.3 + 0.3$$
$$= 0.6$$
$$y(0.6) \approx y_1 = -3.1839$$

for $i = 1$, $x_1 = 0.6$, $y_1 = 4.0448$

$$k_1 = f(x_1, y_1)$$
$$= f(0.6, 4.0448)$$
$$= \frac{1}{3}\left(e^{0.1\times0.6} - 5\sqrt{4.0448}\right)$$
$$= \frac{1}{3}(1.0618 - 10.056)$$
$$= -2.9980$$

$$k_2 = f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1 h\right)$$

$$= f\left(0.6 + \frac{0.3}{2}, 4.0448 + \frac{1}{2}(-2.9980 \times 0.3)\right)$$

$$= f(0.75, 3.5951)$$

$$= \frac{1}{3}\left(e^{0.075} - 5\sqrt{3.5951}\right)$$

$$= -2.8008$$

$$y_2 = y_1 + k_2 h$$

$$= 4.04483 + (-2.8008) \times 0.3$$

$$= 3.2046$$

$$x_2 = x_1 + h$$

$$= 0.6 + 0.3$$

$$= 0.9$$

$$y(0.9) \approx y_2 = 3.2046$$

Thus

$$\frac{dy}{dx}(0.9) = f(x, y)\Big|_{x=0.9}$$

$$\approx f(x_2, y_2)$$

$$= f(0.9, 3.2046)$$

$$= \frac{1}{3}\left(e^{0.1 \times 0.9} - 5\sqrt{3.2046}\right)$$

$$= \frac{1}{3}(1.0942 - 8.9507)$$

$$= -2.6188$$

4. The velocity (m/s) of a body is given as a function of time (seconds) by

$$v(t) = 200\ln(1+t) - t,\ t \ge 0$$

Using the Runge-Kutta 2nd order Ralston method with a step size of 5 seconds, the distance in meters traveled by the body from $t = 2$ to $t = 12$ seconds is estimated most nearly as

(A) 3904.9
(B) 3939.7
(C) 6556.3
(D) 39397

Solution

The correct answer is (A).

$$\frac{dS}{dt} = f(t,S) = 200\ln(1+t) - t,\ t \ge 0$$

In the Ralston method $a_2 = \frac{2}{3}$ is chosen, giving

$$a_1 = \frac{1}{3}$$

$$p_1 = \frac{3}{4}$$

$$q_{11} = \frac{3}{4}$$

resulting in

$$S_{i+1} = S_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

where

$$k_1 = f(t_i, S_i)$$

$$k_2 = f\left(t_i + \frac{3}{4}h, S_i + \frac{3}{4}k_1 h\right)$$

$$h = 5$$

for $i = 0,\ t_0 = 2\,\text{s},\ S_0 = 0\,\text{m}$, we are assuming $S(2) = 0$

$$\begin{aligned} k_1 &= f(t_0, S_0) \\ &= f(2,0) \\ &= 200\ln(1+2) - 2 \\ &= 217.72 \end{aligned}$$

$$k_2 = f\left(t_0 + \frac{3}{4}h, S_0 + \frac{3}{4}k_1 h\right)$$

$$= f\left(2 + \frac{3}{4} \times 5, 0 + \frac{3}{4}(217.72)5\right)$$

$$= f(5.75, 816.46)$$

$$= 200\ln(1 + 5.75) - (5.75)$$

$$= 381.91 - 5.75$$

$$= 376.16$$

$$S_1 = S_0 + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

$$= 0 + \left(\frac{1}{3}(217.72) + \frac{2}{3}(376.16)\right) \times 5$$

$$= 1616.7 \text{ m}$$

$$t_1 = t_0 + h$$

$$= 2 + 5$$

$$= 7$$

$$S(7) \approx S_1 = 1616.7 \text{ m}$$

for $i = 1$, $t_1 = 7\text{ s}$, $S_1 = 1616.7\text{ m}$

$$k_1 = f(t_1, S_1)$$

$$= f(7, 1616.7)$$

$$= 200\ln(1 + 7) - 7$$

$$= 408.89$$

$$k_2 = f\left(t_1 + \frac{3}{4}h, S_1 + \frac{3}{4}k_1 h\right)$$

$$= f\left(7 + \frac{3}{4} \times 5, 1616.7 + \frac{3}{4}(408.89)5\right)$$

$$= f(10.75, 3150.1)$$

$$= 200\ln(1 + 10.75) - (10.75)$$

$$= 492.77 - 10.75$$

$$= 482.02$$

$$S_2 = S_1 + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

$$= 1616.7 + \left(\frac{1}{3}(408.89) + \frac{2}{3}(482.02)\right) \times 5$$

$$= 1616.7 + (457.64) \times 5$$

$$= 3904.9 \text{ m}$$

$$
\begin{aligned}
t_2 &= t_1 + h \\
&= 7 + 5 \\
&= 12
\end{aligned}
$$

$$S(12) \approx S_2 = 3904.9\ \text{m}$$

Hence the distance covered between $t = 2$ and $t = 12$ seconds is

$$
\begin{aligned}
d &= S(12) - S(2) \\
&\approx S_2 - S_0 \\
&= 3904.9 - 0 \\
&= 3904.9\ \text{m}
\end{aligned}
$$

5. The Runge-Kutta 2nd order method can be derived by using the first three terms of the Taylor series of writing the value of y_{i+1} (that is the value of y at x_{i+1}) in terms of y_i (that is the value of y at x_i) and all the derivatives of y at x_i. If $h = x_{i+1} - x_i$, the explicit expression for y_{i+1} if the first three terms of the Taylor series are chosen for solving the ordinary differential equation

$$\frac{dy}{dx} + 5y = 3e^{-2x},\ y(0) = 7$$

would be

(A) $y_{i+1} = y_i + \left(3e^{-2x_i} - 5y_i\right)h + 5\dfrac{h^2}{2}$

(B) $y_{i+1} = y_i + \left(3e^{-2x_i} - 5y_i\right)h + \left(-21e^{-2x_i} + 25y_i\right)\dfrac{h^2}{2}$

(C) $y_{i+1} = y_i + \left(3e^{-2x_i} - 5y_i\right)h + \left(-6e^{-2x_i}\right)\dfrac{h^2}{2}$

(D) $y_{i+1} = y_i + \left(3e^{-2x_i} - 5y_i\right)h + \left(-6e^{-2x_i} + 5\right)\dfrac{h^2}{2}$

Solution

The correct answer is (B).

The first three terms of the Taylor series are as follows

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2$$

Our ordinary differential equation is rewritten as

$$\frac{dy}{dx} + 5y = 3e^{-2x},\ y(0) = 7$$

$$f(x, y) = 3e^{-2x} - 5y,\ y(0) = 7$$

Now since y is a function of x,

$$\begin{aligned} f'(x, y) &= \frac{\partial f(x, y)}{\partial x} + \frac{\partial f(x, y)}{\partial y}\frac{dy}{dx} \\ &= \frac{\partial}{\partial x}\left(3e^{-2x} - 5y\right) + \frac{\partial}{\partial y}\left[\left(3e^{-2x} - 5y\right)\right]\left(3e^{-2x} - 5y\right) \\ &= -6e^{-2x} + (-5)\left(3e^{-2x} - 5y\right) \\ &= -21e^{-2x} + 25y \end{aligned}$$

The 2nd order formula for the above ordinary differential equation would be

$$\begin{aligned} y_{i+1} &= y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2 \\ &= y_i + \left(3e^{-2x_i} - 5y_i\right)h + \frac{1}{2}\left(-21e^{-2x_i} + 25y_i\right)h^2 \end{aligned}$$

6. A spherical ball is taken out of a furnace at 1200 K and is allowed to cool in air. You are given the following

radius of ball = 2 cm

specific heat of ball = 420 $\dfrac{\text{J}}{\text{kg}\cdot\text{K}}$

density of ball = 7800 $\dfrac{\text{kg}}{\text{m}^3}$

convection coefficient = 350 $\dfrac{\text{J}}{\text{s}\cdot\text{m}^2\cdot\text{K}}$

ambient temperature = 300 K

The ordinary differential equation that is given for the temperature θ of the ball is

$$\frac{d\theta}{dt} = -2.20673\times10^{-13}\left(\theta^4 - 81\times10^8\right)$$

if only radiation is accounted for. The ordinary differential equation if convection is accounted for in addition to radiation is

(A) $\dfrac{d\theta}{dt} = -2.20673\times10^{-13}\left(\theta^4 - 81\times10^8\right) - 1.6026\times10^{-2}(\theta - 300)$

(B) $\dfrac{d\theta}{dt} = -2.20673\times10^{-13}\left(\theta^4 - 81\times10^8\right) - 4.3982\times10^{-2}(\theta - 300)$

(C) $\dfrac{d\theta}{dt} = -1.6026\times10^{-2}(\theta - 300)$

(D) $\dfrac{d\theta}{dt} = -4.3982\times10^{-2}(\theta - 300)$

Solution

The correct answer is (A).

The rate of heat loss due to convection

Rate of heat loss due to convection = $hA(\theta - \theta_a)$ `

where

h = convection coefficient = 350 $\dfrac{\text{J}}{\text{s}\cdot\text{m}^2\cdot\text{K}}$

A = surface area of the ball, m^2

The energy stored by mass is

Energy stored by mass = $mC\theta$

where

m = mass of the ball , kg

C = specific heat of the ball, $\dfrac{\text{J}}{\text{kg}\cdot\text{K}}$

$$A = 4\pi r^2$$
$$= 4\pi \times 0.02^2$$
$$= 0.0050265 \text{ m}^2$$
$$m = \rho V$$
$$= \rho\left(\frac{4}{3}\pi r^3\right)$$
$$= 7800\left(\frac{4}{3}\pi \times 0.02^3\right)$$
$$= 0.26138 \text{ kg}$$

From the energy balance

(Rate at which heat is gained) – (Rate at which heat is lost) = (Rate at which heat is stored)

we get

$$mC\frac{d\theta}{dt} = mC\left(-2.20673 \times 10^{-13}\left(\theta^4 - 81 \times 10^8\right)\right) - hA(\theta - \theta_a)$$
$$\frac{d\theta}{dt} = \left(-2.20673 \times 10^{-13}\left(\theta^4 - 81 \times 10^8\right)\right) - \frac{hA(\theta - \theta_a)}{mC}$$
$$= \left(-2.20673 \times 10^{-13}\left(\theta^4 - 81 \times 10^8\right)\right) - \frac{(350)(0.0050265)(\theta - 300)}{0.26138 \times 420}$$
$$= -2.20673 \times 10^{-13}\left(\theta^4 - 81 \times 10^8\right) - 0.016026(\theta - 300)$$
$$= -2.20673 \times 10^{-13}\left(\theta^4 - 81 \times 10^8\right) - 1.6026 \times 10^{-2}(\theta - 300)$$