Holistic Numerical Methods Institute

committed to bringing numerical methods to undergraduates

# Multiple-Choice Test
# Euler's Method
# Ordinary Differential Equations
# COMPLETE SOLUTION SET

1. To solve the ordinary differential equation

$$3\frac{dy}{dx}+5y^2=\sin x,\ y(0)=5$$

by Euler's method, you need to rewrite the equation as

(A) $\frac{dy}{dx}=\sin x-5y^2,\ y(0)=5$

(B) $\frac{dy}{dx}=\frac{1}{3}\left(\sin x-5y^2\right),\ y(0)=5$

(C) $\frac{dy}{dx}=\frac{1}{3}\left(-\cos x-\frac{5y^3}{3}\right),\ y(0)=5$

(D) $\frac{dy}{dx}=\frac{1}{3}\sin x,\ y(0)=5$

**Solution**

*The correct answer is (B).*

To solve ordinary differential equations by Euler's method, you need to rewrite the equation in the following form

$$\frac{dy}{dx}=f(x,y),\ y(0)=y_0$$

Thus,

$$3\frac{dy}{dx}+5y^2=\sin x,\ y(0)=5$$

$$3\frac{dy}{dx}=\sin x-5y^2,\ y(0)=5$$

$$\frac{dy}{dx}=\frac{1}{3}\left(\sin x-5y^2\right),\ y(0)=5$$

2. Given

$$3\frac{dy}{dx}+5y^2=\sin x,\ y(0.3)=5$$

and using a step size of $h=0.3$, the value of $y(0.9)$ using Euler's method is most nearly

(A) $-35.318$
(B) $-36.458$
(C) $-658.91$
(D) $-669.05$

**Solution**

*The correct answer is (A).*

First rewrite the differential equation in the proper form.

$$\frac{dy}{dx}=\frac{1}{3}\left(\sin x-5y^2\right)$$

$$f(x,y)=\frac{1}{3}\left(\sin x-5y^2\right)$$

Euler's method is given by

$$y_{i+1}=y_i+f(x_i,y_i)h$$

where

$$h=0.3$$

For $i=0$, $x_0=0.3$, $y_0=5$

$$\begin{aligned}y_1&=y_0+f(x_0,y_0)h\\&=5+f(0.3,5)\times 0.3\\&=5+\frac{1}{3}\left(\sin(0.3)-5(5)^2\right)\times 0.3\\&=5+(-12.470)\\&=-7.4704\end{aligned}$$

$y_1$ is the approximate value of $y$ at

$$x=x_1=x_0+h=0.3+0.3=0.6$$

For $i=1$, $x_1=0.6$, $y_1=-7.4704$

$$\begin{aligned}y_2&=y_1+f(x_1,y_1)h\\&=-7.4704+f(0.6,-7.4704)\times 0.3\\&=-7.4704+\frac{1}{3}\left(\sin(0.6)-5(-7.4704)^2\right)\times 0.3\\&=-7.4704-27.847\\&=-35.318\end{aligned}$$

$y_2$ is the approximate value of $y$ at

$$x=x_2=x_1+h=0.6+0.3=0.9$$

$$y(0.9)\approx -35.318$$

3. Given

$$3\frac{dy}{dx}+\sqrt{y}=e^{0.1x},\ y(0.3)=5$$

and using a step size of $h=0.3$, the best estimate of $\frac{dy}{dx}(0.9)$ using Euler's method is most nearly

(A) $-0.37319$
(B) $-0.36288$
(C) $-0.35381$
(D) $-0.34341$

**Solution**
*The correct answer is (B).*

First rewrite the differential equation in the proper form.

$$\frac{dy}{dx}=\frac{1}{3}\left(e^{0.1x}-\sqrt{y}\right)$$

$$f(x,y)=\frac{1}{3}\left(e^{0.1x}-\sqrt{y}\right)$$

Euler's method is given by

$$y_{i+1}=y_i+f(x_i,y_i)h$$

where

$$h=0.3$$

For $i=0$, $x_0=0.3$, $y_0=5$

$$\begin{aligned}y_1&=y_0+f(x_0,y_0)h\\&=5+f(0.3,5)\times 0.3\\&=5+\frac{1}{3}\left(e^{0.1\times 0.3}-\sqrt{5}\right)\times 0.3\\&=5+(-0.12056)\\&=4.8794\end{aligned}$$

$y_1$ is the approximate value of $y$ at

$$x=x_1=x_0+h=0.3+0.3=0.6$$

For $i=1$, $x_1=0.6$, $y_1=4.8794$

$$\begin{aligned}y_2&=y_1+f(x_1,y_1)h\\&=4.8794+f(0.6,4.8794)\times 0.3\\&=4.8794+\frac{1}{3}\left(e^{0.1\times 0.6}-\sqrt{4.8794}\right)\times 0.3\\&=4.8794+(-0.11471)\end{aligned}$$

$y_2$ is the approximate value of $y$ at

$$x=x_2=x_1+h=0.6+0.3=0.9$$

$$y(0.9) \approx 4.7647$$

Thus

$$\frac{dy}{dx} = \frac{1}{3}\left(e^{0.1x} - \sqrt{y}\right)$$

$$\frac{dy}{dx}(0.9) \approx \frac{1}{3}\left(e^{0.1\times 0.9} - \sqrt{4.7647}\right)$$

$$= -0.36288$$

4. The velocity (m/s) of a body is given as a function of time (seconds) by

$$v(t) = 200\ln(1+t) - t, t \geq 0$$

Using Euler's method with a step size of 5 seconds, the distance traveled in meters by the body from $t = 2$ to $t = 12$ seconds is most nearly

(A) 3133.1
(B) 3939.7
(C) 5638.0
(D) 39397

**Solution**

*The correct answer is (A).*

$$v(t) = 200\ln(1+t) - t$$

$$\frac{dS}{dt} = 200\ln(1+t) - t$$

$$f(t, S) = 200\ln(1+t) - t$$

Euler's method is given by

$$S_{i+1} = S_i + f(t_i, S_i)h$$

where

$$h = 0.5$$

For $i = 0$, $t_0 = 2$ s, $S_0 = 0$ m (assuming $S_0 = 0$ m would make $S_2$ the value of the distance covered, as the distance covered is $S_2 - S_0$)

$$\begin{aligned} S_1 &= S_0 + f(t_0, S_0) \times h \\ &= 0 + f(2,0) \times 5 \\ &= 0 + (200\ln(1+2) - 2) \times 5 \\ &= 1088.6 \text{ m} \end{aligned}$$

$$\begin{aligned} t_1 &= t_0 + h \\ &= 2 + 5 \\ &= 7 \end{aligned}$$

For $i = 1$, $t_1 = 7$ s, $S_1 = 1088.61$ m

$$\begin{aligned} S_2 &= S_1 + f(t_1, S_1) \times h \\ &= 1088.6 + f(7, 1088.6) \times 5 \\ &= 1088.6 + (200\ln(1+7) - 7) \times 5 \\ &= 1088.6 + 2044.4 \\ &= 3133.1 \text{ m} \end{aligned}$$

$$\begin{aligned} S(12) - S(2) &\approx S_2 - S_0 \\ &= 3133.1 \text{ m} \end{aligned}$$

**Note to the student:**

You do not have to assume $S_0 = 0$ m. Instead, let it be some unknown constant, that is, $S_0 = C$. In that case, if you follow Euler's method as above, you would get

$$\begin{aligned} S_1 &= S_0 + f(t_0, S_0) \times h \\ &= C + f(2,0) \times 5 \\ &= C + (200\ln(1+2) - 2) \times 5 \\ &= C + 1088.6 \text{ m} \end{aligned}$$

$$\begin{aligned} t_1 &= t_0 + h \\ &= 2 + 5 \\ &= 7 \end{aligned}$$

For $i = 1$, $t_1 = 7\,\text{s}$, $S_1 = C + 1088.61$ m

$$\begin{aligned} S_2 &= S_1 + f(t_1, S_1) \times h \\ &= C + 1088.6 + f(7, 1088.6) \times 5 \\ &= C + 1088.6 + (200\ln(1+7) - 7) \times 5 \\ &= C + 1088.6 + 2044.4 \\ &= C + 3133.1 \text{ m} \end{aligned}$$

$$\begin{aligned} S(12) - S(2) &\approx S_2 - S_0 \\ &= C + 3133.1 - C \\ &= 3133.1 \text{ m} \end{aligned}$$

5. Euler's method can be derived by using the first two terms of the Taylor series of writing the value of $y_{i+1}$, that is the value of $y$ at $x_{i+1}$, in terms of $y_i$ and all the derivatives of $y$ at $x_i$. If $h = x_{i+1} - x_i$, the explicit expression for $y_{i+1}$ if the first three terms of the Taylor series are chosen for the ordinary differential equation

$$2\frac{dy}{dx} + 3y = e^{-5x},\ y(0) = 7$$

would be

(A) $y_{i+1} = y_i + \frac{1}{2}\left(e^{-5x_i} - 3y_i\right)h$

(B) $y_{i+1} = y_i + \frac{1}{2}\left(e^{-5x_i} - 3y_i\right)h - \frac{1}{2}\left(\frac{5}{2}e^{-5x_i}\right)h^2$

(C) $y_{i+1} = y_i + \frac{1}{2}\left(e^{-5x_i} - 3y_i\right)h + \frac{1}{2}\left(-\frac{13}{4}e^{-5x_i} + \frac{9}{4}y_i\right)h^2$

(D) $y_{i+1} = y_i + \frac{1}{2}\left(e^{-5x_i} - 3y_i\right)h - \frac{3}{2}y_i h^2$

**Solution**

*The correct answer is (C).*

The differential equation

$$2\frac{dy}{dx} + 3y = e^{-5x},\ y(0) = 7$$

is rewritten as

$$\frac{dy}{dx} = \frac{1}{2}\left(e^{-5x} - 3y\right),\ y(0) = 7$$

$$f(x, y) = \frac{1}{2}\left(e^{-5x} - 3y\right)$$

The Taylor series is given by

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i, y_i}(x_{i+1} - x_i) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_i, y_i}(x_{i+1} - x_i)^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_i, y_i}(x_{i+1} - x_i)^3 + \ldots$$

$$y_{i+1} = y_i + f(x_i, y_i)(x_{i+1} - x_i) + \frac{1}{2!}f'(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!}f''(x_i, y_i)(x_{i+1} - x_i)^3 + \ldots$$

If we look at the first three terms of the Taylor series

$$y_{i+1} = y_i + f(x_i, y_i)(x_{i+1} - x_i) + \frac{1}{2!}f'(x_i, y_i)(x_{i+1} - x_i)^2$$

$$= y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2$$

where

$$h = x_{i+1} - x_i$$

$$
\begin{aligned}
f'(x,y) &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \\
&= \frac{1}{2}\left(-5e^{-5x}\right) + \left(-\frac{3}{2}\right)\left(\frac{1}{2}\left(e^{-5x} - 3y\right)\right) \\
&= -\frac{13}{4}e^{-5x} + \frac{9}{4}y
\end{aligned}
$$

then the value of $y_{i+1}$ is given by

$$
y_{i+1} = y_i + \frac{1}{2}\left(e^{-5x_i} - 3y_i\right)h + \frac{1}{2}\left(-\frac{13}{4}e^{-5x_i} + \frac{9}{4}y_i\right)h^2
$$

6. A homicide victim is found at 6:00 PM in an office building that is maintained at 72 °F. When the victim was found, his body temperature was at 85 °F. Three hours later at 9:00 PM, his body temperature was recorded at 78 °F. Assume the temperature of the body at the time of death is the normal human body temperature of 98.6 °F.

The governing equation for the temperature $\theta$ of the body is

$$\frac{d\theta}{dt} = -k(\theta - \theta_a)$$

where

$\theta$ = temperature of the body, °F

$\theta_a$ = ambient temperature, °F

$t$ = time, hours

$k$ = constant based on thermal properties of the body and air

The estimated time of death most nearly is

(A) 2:11 PM
(B) 3:13 PM
(C) 4:34 PM
(D) 5:12 PM

**Solution**

*The correct answer is (B).*

$$\frac{d\theta}{dt} = -k(\theta - \theta_a)$$

$$\frac{d\theta}{dt} + k\theta = k\theta_a$$

The characteristic equation of the above ordinary differential equations is

$$r + k = 0$$

The solution to this equation is

$$r = -k$$

$$\theta_H = Ae^{-kt}$$

$$(D + k)\theta = k\theta_a$$

The particular solution is of the form

$$\theta_P = B$$

Substituting this solution in the ordinary differential equation,

$$0 + kB = k\theta_a$$

$$B = \theta_a$$

The complete solution is

$$\theta = \theta_H + \theta_P$$

$$= Ae^{-kt} + \theta_a$$

Given

$$\theta_a = 72$$

and using 12 noon as the reference time of $t = 0$,

$\theta(6) = 85$
$\theta(9) = 78$
$\theta(B) = 98.6$
where
$B$ = time of death
we get

$$85 = Ae^{-k \times 6} + 72 \quad (1)$$
$$78 = Ae^{-k \times 9} + 72 \quad (2)$$
$$98.6 = Ae^{-k \times B} + 72 \quad (3)$$

Use Equations (1) and (2) to find $A$ and $k$.

$$85 = Ae^{-k \times 6} + 72$$
$$Ae^{-k \times 6} = 13 \quad (4)$$
$$78 = Ae^{-k \times 9} + 72$$
$$Ae^{-k \times 9} = 6 \quad (5)$$

Dividing Equation (4) by Equation (5) gives

$$\frac{Ae^{-k6}}{Ae^{-k9}} = \frac{13}{6}$$
$$e^{3k} = 2.1667$$
$$k = \frac{1}{3}(\ln(2.1667))$$
$$= 0.25773 \frac{1}{\text{hours}}$$

Knowing the value of $k$, from Equation (5)

$$A = 61.028\ °\text{F}$$

Substitute $k$ and $A$ into Equation (3) to find $B$.

$$98.6 = Ae^{-k \times B} + 72$$
$$98.6 = 61.028e^{-0.25773 \times B} + 72$$
$$26.6 = 61.028e^{-0.25773 \times B}$$
$$\ln 26.6 = \ln 61.028 + (-0.25773B)$$
$$0.25773B = 0.83042$$
$$B = 3.2220 \text{ hours}$$

**Note to the student:**
You can also do the problem by assuming that the initial time reference is zero, and that the temperature then is $\theta(0) = 98.6$. Then the temperature is given at the time the body was found as $\theta(C) = 85$ °F, and that $\theta(C + 3) = 78$ °F. You can now find $k$, $A$ and $C$ just like as given above. The value of $C$ in fact is the time between the body was found and the time of death. You will get $C = 2.7780$ hrs.

The time of death is 3.2220 hrs from 12 noon, that is $3 : (0.2220 \times 60)$ PM = 3:13 PM.