Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
Nonlinear Regression
Regression
COMPLETE SOLUTION SET

1. When using the transformed data model to find the constants of the regression model $y = ae^{bx}$ to best fit $(x_1, y_1), (x_2, y_2), ... (x_n, y_n)$, the sum of the square of the residuals that is minimized is

(A) $\sum_{i=1}^{n}\left(y_i - ae^{bx_i}\right)^2$

(B) $\sum_{i=1}^{n}\left(\ln(y_i) - \ln(a) - bx_i\right)^2$

(C) $\sum_{i=1}^{n}\left(y_i - \ln(a) - bx_i\right)^2$

(D) $\sum_{i=1}^{n}\left(\ln(y_i) - \ln(a) - b\ln(x_i)\right)^2$

Solution
The correct answer is (B).

Taking the natural log of both sides of the regression model

$$y = ae^{bx}$$

gives

$$\ln(y) = \ln(a) + bx$$

The residual at each data point x_i is

$$E_i = \ln(y_i) - \ln(a) - bx_i$$

The sum of the square of the residuals for the transformed data is

$$S_r = \sum_{i=1}^{n} E_i^2$$

$$= \sum_{i=1}^{n}\left(\ln(y_i) - \ln(a) - bx_i\right)^2$$

2. It is suspected from theoretical considerations that the rate of water flow from a firehouse is proportional to some power of the nozzle pressure. Assume pressure data is more accurate. You are transforming the data.

Flow rate, F (gallons/min)	96	129	135	145	168	235
Pressure, p (psi)	11	17	20	25	40	55

The exponent of the nozzle pressure in the regression model $F = ap^b$ most nearly is

(A) 0.49721
(B) 0.55625
(C) 0.57821
(D) 0.67876

Solution

The correct answer is (A).

The transforming of the above data is done as follows.

$$F = ap^b$$
$$\ln(F) = \ln(a) + b\ln(p)$$
$$z = a_0 + bx$$

where

$$z = \ln(F)$$
$$x = \ln(p)$$
$$a_0 = \ln(a)$$

implying

$$a = e^{a_0}$$

There is a linear relationship between z and x.

Linear regression constants are given by

$$b = \frac{n\sum_{i=1}^{n} x_i z_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} z_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$a_0 = \frac{\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} z_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i z_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

Since

$$n = 6$$

$$\sum_{i=1}^{6} x_i z_i = \ln(11)\times\ln(96) + \ln(17)\times\ln(129) + \ln(20)\times\ln(135)$$
$$+ \ln(25)\times\ln(145) + \ln(40)\times\ln(168) + \ln(55)\times\ln(235)$$
$$= 96.208$$

$$\sum_{i=1}^{6} x_i = \ln(11) + \ln(17) + \ln(20) + \ln(25) + \ln(40) + \ln(55) = 19.142$$

$$\sum_{i=1}^{6} z_i = \ln(96) + \ln(129) + \ln(135) + \ln(145) + \ln(168) + \ln(235) = 29.890$$

$$\sum_{i=1}^{6} x_i^2 = (\ln(11))^2 + (\ln(17))^2 + (\ln(20))^2 + (\ln(25))^2 + (\ln(40))^2 + (\ln(55))^2 = 62.779$$

then

$$b = \frac{6\times 96.208 - 19.142\times 29.890}{6\times 62.779 - 19.142^2}$$
$$= \frac{577.25 - 572.15}{376.67 - 366.41}$$
$$= 0.49721$$

Can you now find what a is?

3. The transformed data model for the stress-strain curve $\sigma = k_1 \varepsilon e^{-k_2 \varepsilon}$ for concrete in compression, where σ is the stress and ε is the strain, is

(A) $\ln(\sigma) = \ln(k_1) + \ln(\varepsilon) - k_2 \varepsilon$

(B) $\ln\left(\frac{\sigma}{\varepsilon}\right) = \ln(k_1) - k_2 \varepsilon$

(C) $\ln\left(\frac{\sigma}{\varepsilon}\right) = \ln(k_1) + k_2 \varepsilon$

(D) $\ln(\sigma) = \ln(k_1 \varepsilon) - k_2 \varepsilon$

Solution

The correct answer is (B)

$$\sigma = k_1 \varepsilon e^{-k_2 \varepsilon}$$

The model can be rewritten as

$$\frac{\sigma}{\varepsilon} = k_1 e^{-k_2 \varepsilon}$$

To transform the data, we take the natural log of both sides

$$\begin{aligned} \ln\left(\frac{\sigma}{\varepsilon}\right) &= \ln\left(k_1 e^{-k_2 \varepsilon}\right) \\ &= \ln(k_1) + \ln\left(e^{-k_2 \varepsilon}\right) \\ &= \ln(k_1) - k_2 \varepsilon \end{aligned}$$

4. In nonlinear regression, finding the constants of the model requires solving simultaneous nonlinear equations. However in the exponential model $y = ae^{bx}$ that is best fit to $(x_1, y_1), (x_2, y_2), \ldots\ldots, (x_n, y_n)$, the value of b can be found as a solution of a single nonlinear equation. That nonlinear equation is given by

(A) $\sum_{i=1}^{n} y_i x_i e^{bx_i} - \sum_{i=1}^{n} y_i e^{bx_i} \sum_{i=1}^{n} x_i = 0$

(B) $\sum_{i=1}^{n} y_i x_i e^{bx_i} - \frac{\sum_{i=1}^{n} y_i e^{bx_i}}{\sum_{i=1}^{n} e^{2bx_i}} \sum_{i=1}^{n} x_i e^{2bx_i} = 0$

(C) $\sum_{i=1}^{n} y_i x_i e^{bx_i} - \frac{\sum_{i=1}^{n} y_i e^{bx_i}}{\sum_{i=1}^{n} e^{2bx_i}} \sum_{i=1}^{n} e^{bx_i} = 0$

(D) $\sum_{i=1}^{n} y_i e^{bx_i} - \frac{\sum_{i=1}^{n} y_i e^{bx_i}}{\sum_{i=1}^{n} e^{2bx_i}} \sum_{i=1}^{n} x_i e^{2bx_i} = 0$

Solution

The correct answer is (B).

Given $(x_1, y_1), (x_2, y_2), \ldots\ldots, (x_n, y_n)$, best fit $y = ae^{bx}$ to the data. The variables a and b are the constants of the exponential model. The residual at each data point x_i is

$$E_i = y_i - ae^{bx_i} \tag{1}$$

The sum of the square of the residuals is

$$S_r = \sum_{i=1}^{n} E_i^2$$

$$= \sum_{i=1}^{n} \left(y_i - ae^{bx_i}\right)^2 \tag{2}$$

To find the constants a and b of the exponential model, we find where S_r is a local minimum or maximum by differentiating with respect to a and b and equating the resulting equations to zero.

$$\frac{\partial S_r}{\partial a} = \sum_{i=1}^{n} 2\left(y_i - ae^{bx_i}\right)\left(-e^{bx_i}\right) = 0$$

$$\frac{\partial S_r}{\partial b} = \sum_{i=1}^{n} 2\left(y_i - ae^{bx_i}\right)\left(-ax_i e^{bx_i}\right) = 0 \tag{3a,b}$$

or

$$-\sum_{i=1}^{n} y_i e^{bx_i} + a\sum_{i=1}^{n} e^{2bx_i} = 0$$

$$\sum_{i=1}^{n} y_i x_i e^{bx_i} - a\sum_{i=1}^{n} x_i e^{2bx_i} = 0 \qquad \text{(4a,b)}$$

Equations (4a) and (4b) are simultaneous nonlinear equations with constants a and b. This is unlike linear regression where the equations to find the constants of the model are simultaneous but linear. In general, iterative methods (such as the Gauss-Newton iteration method, Method of Steepest Descent, Marquardt's Method, Direct search, etc) must be used to find values of a and b.

However, in this case, from Equation (4a), a can be written explicitly in terms of b as

$$a = \frac{\sum_{i=1}^{n} y_i e^{bx_i}}{\sum_{i=1}^{n} e^{2bx_i}} \qquad (5)$$

Substituting Equation (5) in (4b) gives

$$\sum_{i=1}^{n} y_i x_i e^{bx_i} - \frac{\sum_{i=1}^{n} y_i e^{bx_i}}{\sum_{i=1}^{n} e^{2bx_i}} \sum_{i=1}^{n} x_i e^{2bx_i} = 0$$

This equation is still a nonlinear equation in terms of b, and can be solved best by numerical methods such as the bisection method or the secant method.

You can now show that these values of of *a* and *b,* correspond to a local minimum, and since the above nonlinear equation has only one real solution, it corresponds to an absolute minimum.

5. There is a functional relationship between the mass density p of air and the altitude h above the sea level.

Altitude above sea level, h (km)	0.32	0.64	1.28	1.60
Mass Density, ρ (kg/m^3)	1.15	1.10	1.05	0.95

In the regression model $\rho = k_1 e^{-k_2 h}$, the constant k_2 is found as $k_2 = 0.1315$. Assuming the mass density of air at the top of the atmosphere is $1/1000^{\text{th}}$ of the mass density of air at sea level. The altitude in kilometers of the top of the atmosphere most nearly is

(A) 46.2
(B) 46.6
(C) 49.7
(D) 52.5

Solution

The correct answer is (D).

Note to the student: See the alternative answer given later as that is quite a bit shorter.

Since

$$k_2 = 0.1315$$

is given, the sum of the square of the residual is

$$S_r = \sum_{i=1}^{n} \left(\rho_i - k_1 e^{-0.1315 h_i}\right)^2$$

First we need to find the value of the constant k_1.

$$\frac{\partial S_r}{\partial k_1} = \sum_{i=1}^{n} 2\left(\rho_i - k_1 e^{-0.1315 h_i}\right)\left(-e^{-0.1315 h_i}\right) = 0$$

$$-\sum_{i=1}^{n} \rho_i e^{-0.1315 h_i} + k_1 \sum_{i=1}^{n} e^{-2\times 0.1315 h_i} = 0$$

Thus,

$$k_1 = \frac{\sum_{i=1}^{n} \rho_i e^{-0.1315 h_i}}{\sum_{i=1}^{n} e^{-0.263 h_i}}$$

Since

$$n = 4$$

$$\sum_{i=1}^{n} \rho_i e^{-0.1315 h_i} = 1.15 e^{-0.1315\times 0.32} + 1.10 e^{-0.1315\times 0.64} + 1.05 e^{-0.1315\times 1.28} + 0.95 e^{-0.1315\times 1.60}$$

$$= 1.15 \times 0.95879 + 1.10 \times 0.91928 + 1.05 \times 0.84508 + 0.95 \times 0.81026$$

$$= 3.7709$$

$$\sum_{i=1}^{n} e^{-0.263h_i} = e^{-0.263\times 0.32} + e^{-0.263\times 0.64} + e^{-0.263\times 1.28} + e^{-0.263\times 1.60}$$
$$= 0.91928 + 0.84508 + 0.71417 + 0.65652$$
$$= 3.1351$$

the value of the constant k_1 is

$$k_1 = \frac{3.7709}{3.1351}$$
$$= 1.2028$$

Hence

$$\rho = k_1 e^{-k_2 h}$$
$$= 1.2028 e^{-0.1315h} \ \ \mathrm{kg/m^3}$$
$$\rho_{\text{sea-level}} = 1.2028 e^{-0.1315\times 0}$$
$$= 1.2028 \ \ \mathrm{kg/m^3}$$
$$\rho_{\text{top}} = \frac{1}{1000}\rho_{\text{sea-level}}$$
$$= \frac{1}{1000}\times 1.2028$$
$$= 0.0012028 \ \ \mathrm{kg/m^3}$$

$$\rho_{\text{top}} = k_1 e^{-0.1315\times h_{\text{top}}}$$
$$e^{-0.1315\times h_{\text{top}}} = \frac{0.0012028}{1.2028}$$
$$h_{\text{top}} = \frac{\ln(0.001)}{-0.1315}$$
$$= 52.530 \ \ \mathrm{km}$$

Alternative Answer:
Note to the student: Do we really need to find k_1 for this problem?

$$\rho = k_1 e^{-0.1315h}$$
$$\rho_{\text{sea-level}} = k_1 e^{-0.1315\times 0}$$
$$= k_1$$
$$\rho_{top} = k_1 e^{-0.1315 h_{\text{top}}}$$
$$\frac{\rho_{\text{sea-level}}}{\rho_{\text{top}}} = \frac{k_1}{k_1 e^{-0.1315 h_{\text{top}}}}$$

$$\frac{\rho_{\text{sea-level}}}{\frac{1}{1000}\rho_{\text{sea-level}}} = \frac{1}{e^{-0.1315 h_{\text{top}}}}$$

$$\begin{aligned} h_{\text{top}} &= \frac{\ln\left(\frac{1}{1000}\right)}{-0.1315} \\ &= 52.530\,\text{km} \end{aligned}$$

6. A steel cylinder at 80° F of length 12" is placed in a commercially available liquid nitrogen bath $(-315°\text{ F})$. If the thermal expansion coefficient of steel behaves as a second order polynomial function of temperature and the polynomial is found by regressing the data below,

Temperature, T (°F)	Thermal expansion Coefficient, α (μ in/in/°F)
-320	2.76
-240	3.83
-160	4.72
-80	5.43
0	6.00
80	6.47

the reduction in the length of the cylinder in inches most nearly is

(A) 0.0219
(B) 0.0231
(C) 0.0235
(D) 0.0307

Solution
The correct answer is (C).

We are fitting the above data to the following polynomial.

$$\alpha = a_0 + a_1 T + a_2 T^2$$

$$S_r = \sum\left(\alpha_i - a_0 - a_1 T_i - a_2 T_i^2\right)^2$$

There is a quadratic relationship between the thermal expansion coefficient and the temperature, and the coefficients $a_0, a_1,$ and a_2 are found as follows

$$\frac{\partial S_r}{\partial a_0} = \sum_{i=1}^{n} 2\left(\alpha_i - a_0 - a_1 T_i - a_2 T_i^2\right)(-1) = 0$$

$$\frac{\partial S_r}{\partial a_1} = \sum_{i=1}^{n} 2\left(\alpha_i - a_0 - a_1 T_i - a_2 T_i^2\right)(-T_i) = 0$$

$$\frac{\partial S_r}{\partial a_2} = \sum_{i=1}^{n} 2\left(\alpha_i - a_0 - a_1 T_i - a_2 T_i^2\right)(-T_i^2) = 0$$

which gives

$$\begin{bmatrix} n & \left(\sum_{i=1}^{n} T_i\right) & \left(\sum_{i=1}^{n} T_i^2\right) \\ \left(\sum_{i=1}^{n} T_i\right) & \left(\sum_{i=1}^{n} T_i^2\right) & \left(\sum_{i=1}^{n} T_i^3\right) \\ \left(\sum_{i=1}^{n} T_i^2\right) & \left(\sum_{i=1}^{n} T_i^3\right) & \left(\sum_{i=1}^{n} T_i^4\right) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} \alpha_i \\ \sum_{i=1}^{n} T_i \ \alpha_i \\ \sum_{i=1}^{n} T_i^2 \alpha_i \end{bmatrix}$$

Table 1 Summations for calculating constants of model.

i	T (°F)	α (in/in/°F)	T^2	T^3
1	80	6.4700×10^{-6}	6.4000×10^{3}	5.1200×10^{5}
2	0	6.0000×10^{-6}	0.0000	0.0000
3	-80	5.4300×10^{-6}	6.4000×10^{3}	-5.1200×10^{5}
4	-160	4.7200×10^{-6}	2.5600×10^{4}	-4.0960×10^{6}
5	-240	3.8300×10^{-6}	5.7600×10^{4}	-1.3824×10^{7}
6	-320	2.7600×10^{-6}	1.0240×10^{5}	-3.2768×10^{7}
$\sum_{i=1}^{6}$	-7.2000×10^{2}	2.9210×10^{-5}	1.9840×10^{5}	-5.0688×10^{7}

Table 1 (cont)

i	T^4	$T\times\alpha$	$T^2\times\alpha$
1	4.0960×10^{7}	5.1760×10^{-4}	4.1408×10^{-2}
2	0.0000	0.0000	0.0000
3	4.0960×10^{7}	-4.3440×10^{-4}	3.4752×10^{-2}
4	6.5536×10^{8}	-7.5520×10^{-4}	1.2083×10^{-1}
5	3.3178×10^{9}	-9.1920×10^{-4}	2.2061×10^{-1}
6	1.0486×10^{10}	-8.8320×10^{-4}	2.8262×10^{-1}
$\sum_{i=1}^{6}$	1.4541×10^{10}	-2.4744×10^{-3}	7.0022×10^{-1}

We have

$$\begin{bmatrix} 6 & -7.2000\times10^{2} & 1.9840\times10^{5} \\ -7.2000\times10^{2} & 1.9840\times10^{5} & -5.0688\times10^{7} \\ 1.9840\times10^{5} & -5.0688\times10^{7} & 1.4541\times10^{10} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2.9210\times10^{-5} \\ -2.4744\times10^{-3} \\ 7.0022\times10^{-1} \end{bmatrix}$$

Solving the above system of simultaneous linear equations, we get

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 6.0238\times10^{-6} \\ 6.3319\times10^{-9} \\ -1.1965\times10^{-11} \end{bmatrix}$$

The polynomial regression model is

$$\begin{aligned} \alpha &= a_0 + a_1 T + a_2 T^2 \\ &= 6.0237\times10^{-6} + 6.3375\times10^{-9}\,T - 1.1942\times10^{-11}\,T^2 \end{aligned}$$

Since

$$\begin{aligned} \Delta L &= L_0 \times \int_{T_{\text{room}}}^{T_{\text{fluid}}} \alpha\, dT \\ &= 12\times\int_{80}^{-315}\left(6.0237\times10^{-6} + 6.3375\times10^{-9}\,T - 1.1942\times10^{-11}\,T^2\right)dT \\ &= 12\times\left[6.0237\times10^{-6}\,T + \frac{6.3375\times10^{-9}}{2}T^2 - \frac{1.1942\times10^{-11}}{3}T^3\right]_{80}^{-315} \\ &= 12\times\left[6.0237\times10^{-6}\,T + 3.1687\times10^{-9}\,T^2 - 3.9807\times10^{-12}\,T^3\right]_{80}^{-315} \\ &= 12\times\left[6.0237\times10^{-6}(-315) + 3.1687\times10^{-9}(-315^2) - 3.9807\times10^{-12}(-315^3)\right] \\ &\quad -12\times\left[6.0237\times10^{-6}(80) + 3.1687\times10^{-9}(80^2) - 3.9807\times10^{-12}(80^3)\right] \\ &= 12\times\left[-1.4586\times10^{-3} - 5.0014\times10^{-4}\right] \\ &= 0.023505'' \end{aligned}$$