Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
Spline Method
Interpolation
COMPLETE SOLUTION SET

1. The following *n* data points, (x_1, y_1), (x_2, y_2), …….. (x_n, y_n), are given. For conducting quadratic spline interpolation the *x*-data needs to be
   (A) equally spaced
   (B) placed in ascending or descending order of x-values
   (C) integers
   (D) positive

Solution
The correct answer is (B).

The following *n* data points, (x_1, y_1), (x_2, y_2), …….. (x_n, y_n), are given. For conducting quadratic spline interpolation the *x*-data needs to be arranged in ascending or descending order.

2. In cubic spline interpolation,

(A) the first derivatives of the splines are continuous at the interior data points

(B) the second derivatives of the splines are continuous at the interior data points

(C) the first and the second derivatives of the splines are continuous at the interior data points

(D) the third derivatives of the splines are continuous at the interior data points

Solution

The correct answer is (C).

In cubic spline interpolation, the first and the second derivatives of the splines are continuous at the interior data points. In quadratic spline interpolation, only the first derivatives of the splines are continuous at the interior data points.

3. The following incomplete y vs. x data is given.

x	1	2	4	6	7
y	5	11	????	????	32

The data is fit by quadratic spline interpolants given by

$$f(x) = ax - 1,\quad 1 \le x \le 2$$
$$f(x) = -2x^2 + 14x - 9,\quad 2 \le x \le 4$$
$$f(x) = bx^2 + cx + d,\quad 4 \le x \le 6$$
$$f(x) = 25x^2 - 303x + 928,\quad 6 \le x \le 7$$

where a, b, c, and d are constants. The value of c is most nearly

(A) -303.00
(B) -144.50
(C) 0.0000
(D) 14.000

Solution
The correct answer is (C).

Solution Method 1:
Since the first derivatives of two quadratic splines are continuous at the interior points, at $x = 4$

$$\left.\frac{d}{dx}\left(bx^2 + cx - d\right)\right|_{x=4} = \left.\frac{d}{dx}\left(-2x^2 + 14x - 9\right)\right|_{x=4}$$
$$\left.2bx + c\right|_{x=4} = \left.-4x + 14\right|_{x=4}$$
$$2(4)b + c = -4(4) + 14$$
$$8b + c = -2 \qquad (1)$$

and at $x = 6$

$$\left.\frac{d}{dx}\left(bx^2 + cx - d\right)\right|_{x=6} = \left.\frac{d}{dx}\left(25x^2 - 303x + 928\right)\right|_{x=6}$$
$$\left.2bx + c\right|_{x=6} = \left.50x - 303\right|_{x=6}$$
$$2(6)b + c = 50(6) - 303$$
$$12b + c = -3 \qquad (2)$$

Equations (1) and (2) in matrix form

$$\begin{bmatrix} 8 & 1 \\ 12 & 1 \end{bmatrix}\begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

Solving these equations gives

$$b = -0.25$$
$$c = 0$$

Solution Method 2:

The third spline $bx^2 + cx + d$ goes through $x = 4$. However, so does the second spline. We can use this latter knowledge to find the value of y at $x = 4$.

$$f(x) = -2x^2 + 14x - 9, \;\; 2 \le x \le 4$$

$$f(4) = -2(4)^2 + 14(4) - 9$$
$$= 15$$

The third spline $bx^2 + cx + d$ goes through $x = 4$.

Hence

$$b(4)^2 + c(4) + d = 15$$

$$16b + 4c + d = 15 \qquad (1)$$

The third spline $bx^2 + cx + d$ goes through $x = 6$. However, so does the fourth spline. We can use this latter knowledge to find the value of y at $x = 6$.

$$f(x) = 25x^2 - 303x + 928, \;\;\; 6 \le x \le 7$$

$$f(6) = 25(6)^2 - 303(6) + 928$$
$$= 10$$

The third spline $bx^2 + cx + d$ goes through $x = 6$.

Hence

$$b(6)^2 + c(6) + d = 10$$

$$36b + 6c + d = 10 \qquad (2)$$

Since the first derivatives of second and third quadratic splines are continuous at the interior points, at $x = 4$

$$\frac{d}{dx}\left(bx^2 + cx - d\right)\Big|_{x=4} = \frac{d}{dx}\left(-2x^2 + 14x - 9\right)\Big|_{x=4}$$

$$2bx + c\big|_{x=4} = -4x + 14\big|_{x=4}$$

$$2(4)b + c = -4(4) + 14$$

$$8b + c = -2 \qquad (3)$$

Equations (1), (2) and (3) are then

$$16b + 4c + d = 15$$
$$36b + 6c + d = 10$$
$$8b + c + 0d = -2$$

Putting these equations in matrix form gives

$$\begin{bmatrix} 16 & 4 & 1 \\ 36 & 6 & 1 \\ 8 & 1 & 0 \end{bmatrix} \begin{bmatrix} b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \\ 2 \end{bmatrix}$$

Solving the above equations gives

$$b = -0.25$$

$$c = 0$$

$$d = 19$$

4. The following incomplete y vs. x data is given.

x	1	2	4	6	7
y	5	11	????	????	32

The data is fit by quadratic spline interpolants given by

$$f(x) = ax - 1, \quad 1 \le x \le 2,$$
$$f(x) = -2x^2 + 14x - 9, \quad 2 \le x \le 4$$
$$f(x) = bx^2 + cx + d, \quad 4 \le x \le 6$$
$$f(x) = ex^2 + fx + g, \quad 6 \le x \le 7$$

where a, b, c, d, e, f, and g are constants. The value of $\frac{df}{dx}$ at $x = 2.6$ most nearly is

(A) -144.50
(B) -4.0000
(C) 3.6000
(D) 12.200

Solution
The correct answer is (C).

Since the spline

$$f(x) = -2x^2 + 14x - 9$$

is valid in the interval $2 \le x \le 4$, the derivative at $x = 2.6$ is

$$\frac{df}{dx}(x) = -4x + 14$$

$$\begin{aligned}\frac{df}{dx}(2.6) &= -4 \times 2.6 + 14 \\ &= -10.4 + 14 \\ &= 3.6000\end{aligned}$$

5. The following incomplete y vs. x data is given.

x	1	2	4	6	7
y	5	11	????	????	32

The data is fit by quadratic spline interpolants given by

$$f(x) = ax - 1,\quad 1 \le x \le 2,$$
$$f(x) = -2x^2 + 14x - 9,\quad 2 \le x \le 4$$
$$f(x) = bx^2 + cx + d,\quad 4 \le x \le 6$$
$$f(x) = 25x^2 - 303x + 928,\quad 6 \le x \le 7$$

Where $a, b, c,$ and d are constants. What is the value of $\int_{1.5}^{3.5} f(x)dx$?

(A) 23.500
(B) 25.667
(C) 25.750
(D) 28.000

Solution

The correct answer is (C).

To find $\int_{1.5}^{3.5} f(x)dx$ we must take $\int_{1.5}^{2}(ax-1)dx + \int_{2}^{3.5}\left(-2x^2+14x-9\right)dx$ but first we have to find the value of the constant a. Since at $x = 2$, $y = 11$

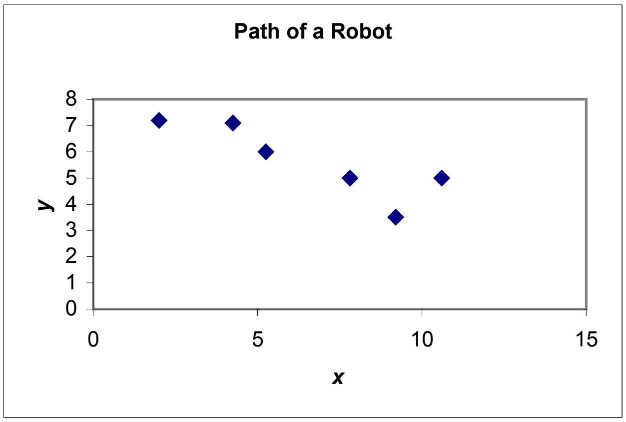
$$a \times 2 - 1 = 11$$
$$a = \frac{11+1}{2}$$
$$= 6$$

Thus,

$$\int_{1.5}^{3.5} f(x)dx = \int_{1.5}^{2}(6x-1)dx + \int_{2}^{3.5}\left(-2x^2+14x-9\right)dx$$
$$= \left[\frac{6}{2}x^2 - x\right]_{1.5}^{2} + \left[-\frac{2}{3}x^3 + \frac{14}{2}x^2 - 9x\right]_{2}^{3.5}$$
$$= \left[\left(3\times 2^2 - 2\right) - \left(3\times 1.5^2 - 1.5\right)\right] +$$
$$\left[\left(-\frac{2}{3}\times 3.5^3 + 7\times 3.5^2 - 9\times 3.5\right) - \left(-\frac{2}{3}\times 2^3 + 7\times 2^2 - 9\times 2\right)\right]$$
$$= \left[(12-2) - (6.75-1.5)\right] +$$
$$\left[(-28.583 + 85.75 - 31.5) - (-5.3333 + 28 - 18)\right]$$
$$= [4.75] + [21]$$
$$= 25.75$$

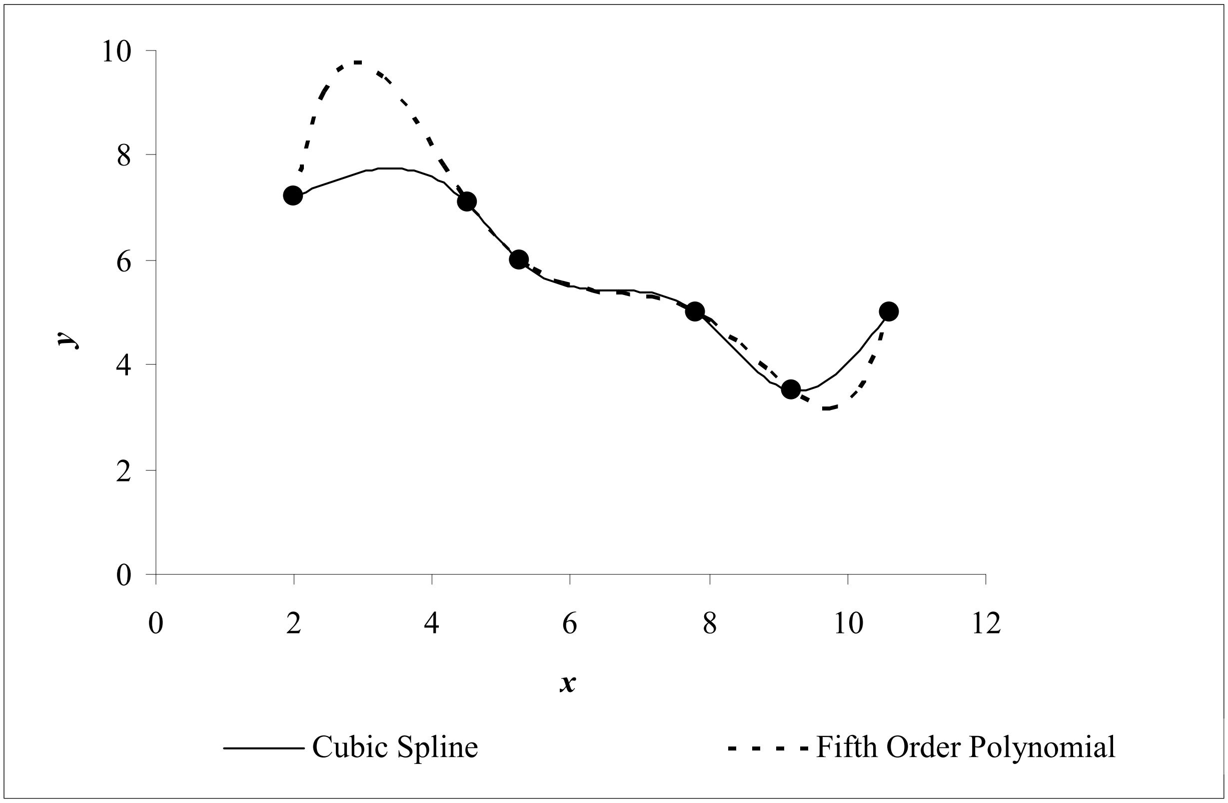
6. A robot needs to follow a path that passes consecutively through six points as shown in the figure. To find the shortest path that is also smooth you would recommend which of the following?

(A) Pass a fifth order polynomial through the data
(B) Pass linear splines through the data
(C) Pass quadratic splines through the data
(D) Regress the data to a second order polynomial



Solution
The correct answer is (C).

Using linear splines (Choice B) would create a straight-line path between consecutive points. Although this will be the shortest path it will not be smooth. Regressing the data to a second order polynomial (Choice D) will result in a smooth path but it will not pass through all the points. As demonstrated in the following figure, using polynomial interpolation such as choice (A) is a bad idea and will result in a long path. By using quadratic spline interpolation (choice C), the path will be short as well as smooth.



10
8
6
4
2
0
y
0
2
4
6
8
10
12
x
Cubic Spline
Fifth Order Polynomial