Multiple-Choice Test
Newton's Divided Difference Polynomial Method
Interpolation
COMPLETE SOLUTION SET

1. If a polynomial of degree n has more than n zeros, then the polynomial is
   (A) oscillatory
   (B) zero everywhere
   (C) quadratic
   (D) not defined

Solution
The correct answer is (B).

A unique polynomial of degree n or less passes through $n+1$ data points. Assume two polynomials $P_n(x)$ and $Q_n(x)$ go through $n+1$ data points,

$$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$$

Then

$$R_n(x) = P_n(x) - Q_n(x)$$

Since $P_n(x)$ and $Q_n(x)$ pass through all the $n+1$ data points,

$$P_n(x_i) = Q_n(x_i), \; i = 0, \ldots, n$$

Hence

$$R_n(x_i) = P_n(x_i) - Q_n(x_i) = 0, \; i = 0, \ldots, n$$

The n^{th} order polynomial $R_n(x)$ has $n+1$ zeros. A polynomial of order n can have $n+1$ zeros only if it is identical to a zero polynomial, that is,

$$R_n(x) \equiv 0$$

Hence

$$P_n(x) \equiv Q_n(x)$$

How can one show that if a second order polynomial has three zeros, then it is zero everywhere?
If $R_2(x) = a_0 + a_1 x + a_2 x^2$, then if it has three zeros at x_1, x_2, and x_3, then

$$R_2(x_1) = a_0 + a_1 x_1 + a_2 x_1^2 = 0$$

$$R_2(x_2) = a_0 + a_1 x_2 + a_2 x_2^2 = 0$$

$$R_2(x_3) = a_0 + a_1 x_3 + a_2 x_3^2 = 0$$

Which in matrix form gives

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The final solution $a_1 = a_2 = a_3 = 0$ exists if the coefficient matrix is invertible. The determinant of the coefficient matrix can be found symbolically with the forward elimination steps of naïve Gauss elimination to give

$$\det\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} = x_2x_3^2 - x_2^2x_3 - x_1x_3^2 + x_1^2x_3 + x_1x_2^2 - x_1^2x_2$$

$$= (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$$

Since

$$x_1 \neq x_2 \neq x_3$$

the determinant is non-zero. Hence, the coefficient matrix is invertible. Therefore, $a_1 = a_2 = a_3 = 0$ is the only solution, that is, $R_2(x) \equiv 0$.

2. The following x - y data is given.

x	15	18	22
y	24	37	25

The Newton's divided difference second order polynomial for the above data is given by

$$f_2(x) = b_0 + b_1(x-15) + b_2(x-15)(x-18)$$

The value of b_1 is most nearly

(A) –1.0480
(B) 0.14333
(C) 4.3333
(D) 24.000

Solution

The correct answer is (C).

Given

$$x_0 = 15$$
$$x_1 = 18$$
$$f_2(x_0) = 24$$
$$f_2(x_1) = 37$$

we have

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

Then

$$\begin{aligned} f_2(x_0) &= b_0 + b_1(x_0 - x_0) + b_2(x_0 - x_0)(x_0 - x_1) \\ &= b_0 \end{aligned}$$

$$\begin{aligned} f_2(x_1) &= b_0 + b_1(x_1 - x_0) + b_2(x_1 - x_0)(x_1 - x_1) \\ &= b_0 + b_1(x_1 - x_0) \\ &= f_2(x_0) + b_1(x_1 - x_0) \end{aligned}$$

Thus,

$$\begin{aligned} b_1 &= \frac{f_2(x_1) - f_2(x_0)}{x_1 - x_0} \\ &= \frac{37 - 24}{18 - 15} \\ &= 4.3333 \end{aligned}$$

3. The polynomial that passes through the following x - y data

x	18	22	24
y	?	25	123

is given by

$$8.125x^2 - 324.75x + 3237, 18 \le x \le 24$$

The corresponding polynomial using Newton's divided difference polynomial is given by

$$f_2(x) = b_0 + b_1(x-18) + b_2(x-18)(x-22)$$

The value of b_2 is

(A) 0.25000
(B) 8.1250
(C) 24.000
(D) not obtainable with the information given

Solution

The correct answer is (B).

Expanding,

$$\begin{aligned} f_2(x) &= b_0 + b_1(x-18) + b_2(x-18)(x-22) \\ &= b_0 + b_1x - 18b_1 + b_2(x^2 - 40x + 396) \\ &= (b_0 - 18b_1 + 396b_2) + (b_1 - 40b_2)x + b_2x^2 \end{aligned}$$

This needs to be the same as

$$8.125x^2 - 324.75x + 3237$$

Hence

$$b_2 = 8.125$$

4. Velocity vs. time data for a body is approximated by a second order Newton's divided difference polynomial as

$$v(t) = b_0 + 39.622(t - 20) + 0.5540(t - 20)(t - 15),\ 10 \le t \le 20$$

The acceleration in $\mathrm{m/s^2}$ at $t = 15$ is

(A) 0.55400
(B) 39.622
(C) 36.852
(D) not obtainable with the given information

Solution

The correct answer is (C).

$$v(t) = b_0 + 39.622(t - 20) + 0.5540(t - 20)(t - 15), \quad 10 \le t \le 20$$

$$\begin{aligned}
a(t) &= \frac{d}{dt}(v(t)) \\
&= 39.622(1) + 0.5540(t - 20) + 0.5540(t - 15) \\
&= 39.622 + 0.5540(t - 20) + 0.5540(t - 15), \quad 10 \le t \le 20 \\
a(15) &= 39.622 + 0.5540(15 - 20) + 0.5540(15 - 15) \\
&= 39.622 - 2.7700 \\
&= 36.852 \text{ m/s}^2
\end{aligned}$$

5. The path that a robot is following on a x-y plane is found by interpolating four data points as

x	2	4.5	5.5	7
y	7.5	7.5	6	5

$$y(x) = 0.15238x^3 - 2.2571x^2 + 9.6048x - 3.9000$$

The length of the path from $x = 2$ to $x = 7$ is

(A) $\sqrt{(7.5-7.5)^2 + (4.5-2)^2} + \sqrt{(6-7.5)^2 + (5.5-4.5)^2} + \sqrt{(5-6)^2 + (7-5.5)^2}$

(B) $\int_2^7 \sqrt{1 + (0.15238x^3 - 2.2571x^2 + 9.6048x - 3.9000)^2}\, dx$

(C) $\int_2^7 \sqrt{1 + (0.45714x^2 - 4.5142x + 9.6048)^2}\, dx$

(D) $\int_2^7 (0.15238x^3 - 2.2571x^2 + 9.6048x - 3.9000) dx$

Solution

The correct answer is (C).

The length S of the curve $y(x)$ from a to b is given by

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

where

$$a = 2$$

$$b = 7$$

giving

$$S = \int_2^7 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$y(x) = 0.15238x^3 - 2.2571x^2 + 9.6048x - 3.9000$$

$$\frac{dy}{dx} = 0.45714x^2 - 4.5142x + 9.6048$$

Thus,

$$S = \int_2^7 \sqrt{1 + \left(0.45714x^2 - 4.5142x + 9.6048\right)^2}\, dx$$

6. The following data of the velocity of a body is given as a function of time.

Time (s)	0	15	18	22	24
Velocity (m/s)	22	24	37	25	123

If you were going to use quadratic interpolation to find the value of the velocity at $t = 14.9$ seconds, the three data points of time you would choose for interpolation are

(A) 0, 15, 18
(B) 15, 18, 22
(C) 0, 15, 22
(D) 0, 18, 24

Solution
The correct answer is (A).

We need to choose the three points closest to $t = 14.9\ \mathrm{s}$ that also bracket $t = 14.9\ \mathrm{s}$. Although the data points in choice (B) are closest to 14.9, they do not bracket it. This would be performing extrapolation, not interpolation. Choices (C) and (D) both bracket $t = 14.9\ \mathrm{s}$ but they are not the closest three data points.

Time (s)	Velocity (m/s)	How far is $t = 14.9\ \mathrm{s}$
0	22	$\lvert 14.9 - 0 \rvert = 14.9$
15	24	$\lvert 14.9 - 15 \rvert = 0.1$
18	37	$\lvert 14.9 - 18 \rvert = 3.1$
22	25	$\lvert 14.9 - 22 \rvert = 7.1$
24	123	$\lvert 14.9 - 24 \rvert = 9.1$