Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
Direct Method
Interpolation
COMPLETE SOLUTION SET

1. Given n+1 data pairs, a unique polynomial of degree ________________ passes through $n+1$ data points.

(A) $n+1$
(B) $n+1$ or less
(C) n
(D) n or less

Solution

The correct answer is (D).

A unique polynomial of degree n or less passes through $n+1$ data points. If the polynomial is not unique, then at least two polynomials of order n or less pass through the $n+1$ data points. Assume two polynomials $P_n(x)$ and $Q_n(x)$ go through $n+1$ data points,

$$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$$

Then

$$R_n(x) = P_n(x) - Q_n(x)$$

Since $P_n(x)$ and $Q_n(x)$ pass through all the $n+1$ data points,

$$P_n(x_i) = Q_n(x_i), i = 0, \ldots, n$$

Hence

$$R_n(x_i) = P_n(x_i) - Q_n(x_i) = 0, i = 0, \ldots, n$$

The n^{th} order polynomial $R_n(x)$ has $n+1$ zeros. A polynomial of order n can have $n+1$ zeros only if it is identical to a zero polynomial, that is,

$$R_n(x) \equiv 0$$

Hence

$$P_n(x) \equiv Q_n(x)$$

Extra Notes for the Student:

How can one show that if a second order polynomial has three zeros, then it is zero everywhere?
If $R_2(x) = a_0 + a_1 x + a_2 x^2$, then if it has three zeros at x_1, x_2, and x_3, then

$$R_2(x_1) = a_0 + a_1 x_1 + a_2 x_1^2 = 0$$

$$R_2(x_2) = a_0 + a_1 x_2 + a_2 x_2^2 = 0$$

$$R_2(x_3) = a_0 + a_1 x_3 + a_2 x_3^2 = 0$$

which in matrix form gives

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The above set of equations has a trivial solution, that is, $a_1 = a_2 = a_3 = 0$. But is this the only solution? That is true if the coefficient matrix is invertible.

The determinant of the coefficient matrix can be found symbolically with the forward elimination steps of naïve Gauss elimination to give

$$\det \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} = x_2 x_3^2 - x_2^2 x_3 - x_1 x_3^2 + x_1^2 x_3 + x_1 x_2^2 - x_1^2 x_2$$

$$= (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$$

Since

$$x_1 \neq x_2 \neq x_3$$

the determinant is non-zero. Hence the coefficient matrix is invertible. $a_1 = a_2 = a_3 = 0$ is the only solution, that is, $R_2(x) \equiv 0$.

2. The data of the velocity of a body as a function of time is given as follows.

Time (s)	0	15	18	22	24
Velocity (m/s)	22	24	37	25	123

The velocity in m/s at $16\,\text{s}$ using linear polynomial interpolation is most nearly

(A) 27.867
(B) 28.333
(C) 30.429
(D) 43.000

Solution

The correct answer is (B).

For the first order polynomial, we choose the velocity given by

$$v(t) = a_0 + a_1 t$$

Since we want to find the velocity at $t = 16$, we choose the two data points that are closest to $t = 16$ and that also bracket $t = 16$. Those two points are $t_0 = 15$ and $t_1 = 18$. Then

$$t_0 = 15, v(t_0) = 24$$

$$t_1 = 18, v(t_1) = 37$$

gives

$$v(15) = a_0 + a_1(15) = 24$$

$$v(18) = a_0 + a_1(18) = 37$$

Writing the equations in matrix form

$$\begin{bmatrix} 1 & 15 \\ 1 & 18 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 24 \\ 37 \end{bmatrix}$$

and solving the above two equations gives

$$a_0 = -41$$

$$a_1 = 4.3333$$

Hence

$$v(t) = a_0 + a_1 t$$

$$= -41 + 4.3333t, \ 15 \le t \le 18$$

$$v(16) = -41 + 4.3333(16)$$

$$= 28.333 \text{ m/s}$$

3. The following data of the velocity of a body as a function of time is given as follows.

Time (s)	0	15	18	22	24
Velocity (m/s)	22	24	37	25	123

The velocity in m/s at $16\,\text{s}$ using quadratic polynomial interpolation is most nearly

(A) 27.867
(B) 28.333
(C) 30.429
(D) 43.000

Solution

The correct answer is (C).

For second order polynomial interpolation (also called quadratic interpolation), we choose the velocity given by

$$v(t) = a_0 + a_1 t + a_2 t^2$$

Since we want to find the velocity at $t = 16$, we need to choose the three data points that are closest to $t = 16$ and that also bracket $t = 16$. These three points are $t_0 = 15$, $t_1 = 18$, and $t_2 = 22$.

$$t_0 = 15,\ v(t_0) = 24$$

$$t_1 = 18,\ v(t_1) = 37$$

$$t_2 = 22,\ v(t_2) = 25$$

gives

$$v(15) = a_0 + a_1(15) + a_2(15)^2 = 24$$

$$v(18) = a_0 + a_1(18) + a_2(18)^2 = 37$$

$$v(22) = a_0 + a_1(22) + a_2(22)^2 = 25$$

Writing the three equations in matrix form

$$\begin{bmatrix} 1 & 15 & 225 \\ 1 & 18 & 324 \\ 1 & 22 & 484 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 24 \\ 37 \\ 25 \end{bmatrix}$$

and the solution of the above three equations gives

$$a_0 = -323.86$$

$$a_1 = 38.905$$

$$a_2 = -1.0476$$

Hence

$$v(t) = -323.86 + 38.905t - 1.0476t^2, \ 15 \le t \le 22$$

At $t = 16$,

$$v(16) = -323.86 + 38.905(16) - 1.0476(16)^2$$

$$= 30.429 \text{ m/s}$$

4. The following data of the velocity of a body is given as a function of time

Time (s)	0	15	18	22	24
Velocity (m/s)	22	24	37	25	123

Using quadratic interpolation, the interpolant

$$v(t) = 8.667t^2 - 349.67t + 3523, 18 \le t \le 24$$

approximates the velocity of the body. From this information, one of the times in seconds at which the velocity of the body is 35 m/s during the above time interval of $t = 18\,\text{s}$ to $t = 24\,\text{s}$ is

(A) 18.667
(B) 20.850
(C) 22.200
(D) 22.294

Solution

The correct answer is (D).

Using the interpolant, set the velocity equal to 35 m/s and solve for time.

$$35 = 8.667t^2 - 349.67t + 3523$$

$$0 = 8.667t^2 - 349.67t + 3488$$

Using the quadratic equation solution

$$t = \frac{-(-349.67) \pm \sqrt{(-349.67)^2 - 4 \times 8.667 \times 3488}}{2 \times 8.667}$$

$$= \frac{349.67 \pm \sqrt{1347.1249}}{17.334}$$

gives

$$t = 22.294\,\text{s}$$

$$t = 18.055\,\text{s}$$

5. The following data of the velocity of a body is given as a function of time

Time (s)	0	15	18	22	24
Velocity (m/s)	22	24	37	25	123

One of the interpolant approximations for the velocity from the above data is given as

$$v(t) = 8.6667t^2 - 349.67t + 3523, 18 \le t \le 24$$

Using the above interpolant, the distance in meters covered by the body between $t = 19\text{s}$ and $t = 22\text{s}$ is most nearly

(A) 10.337
(B) 88.500
(C) 93.000
(D) 168.00

Solution

The correct answer is (A).

Since $v = \frac{dx}{dt}$, taking the integral of the velocity will give the location, x. By taking the integral over the interval of $t = 19\text{s}$ to $t = 22\text{s}$ we can find the distance traveled, s, over that interval.

$$s = \int_{19}^{22} \left(8.6667t^2 - 349.67t + 3523\right) dt$$

$$= \left[2.8889t^3 - 174.84t^2 + 3523t\right]_{19}^{22}$$

$$= \left(2.8889(22)^3 - 174.84(22)^2 + 3523(22)\right) - \left(2.8889(19)^3 - 174.84(19)^2 + 3523(19)\right)$$

$$= 23647 - 23637$$

$$= 10.337 \text{ m}$$

6. The following data of the velocity of a body is given as a function of time.

Time (s)	0	15	18	22	24
Velocity (m/s)	22	24	37	25	123

If you were going to use quadratic interpolation to find the value of the velocity at $t = 14.9$ seconds, what three data points of time would you choose for interpolation?

(A) 0, 15, 18
(B) 15, 18, 22
(C) 0, 15, 22
(D) 0, 18, 24

Solution
The correct answer is (A).

We need to choose the three points closest to $t = 14.9\,\text{s}$ that also bracket $t = 14.9\,\text{s}$. Although the data points in choice (B) are closest to 14.9, they do not bracket it. This would be performing extrapolation, not interpolation. Choices (C) and (D) both bracket $t = 14.9\,\text{s}$ but they are not the closest three data points.

Time (s)	Velocity (m/s)	How far is $t = 14.9\,\text{s}$
0	22	$\lvert 14.9 - 0 \rvert = 14.9$
15	24	$\lvert 14.9 - 15 \rvert = 0.1$
18	37	$\lvert 14.9 - 18 \rvert = 3.1$
22	25	$\lvert 14.9 - 22 \rvert = 7.1$
24	123	$\lvert 14.9 - 24 \rvert = 9.1$