Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
Length of Curve

COMPLETE SOLUTION SET

1. The arc length of a smooth cartesian curve $f(x)$ from a to b is given by

(A) $\int_a^b f(x)dx$

(B) $(b-a)^2+(f(b)-f(a))^2$

(C) $\int_a^b \sqrt{\left(1+\frac{dy}{dx}\right)}\,dx$

(D) $\int_a^b \sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)}\,dx$

Solution

The correct answer is (D)

The arc length of a smooth continuous curve $f(x)$ from a to b is given by:

$$L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

You can appreciate this by expanding the formula as follows

$$L=\int_a^b \sqrt{\frac{(dx)^2+(dy)^2}{(dx)^2}}\,dx$$

$$=\int_a^b \sqrt{(dx)^2+(dy)^2}\,dx$$

$$=\int_a^b ds$$

The above is the integral of the differential length of an arc.

2. Which of these definite integrals gives the length of the arc of the function

$f(x) = 3x + \sin x$ from $x = 2$ to $x = 5$

(A) $\int_2^5 (3x + \sin x)dx$

(B) $\int_2^5 \sqrt{1 + (3 + \cos x)}dx$

(C) $\int_2^5 \sqrt{1 + (3 + \cos x)^2}dx$

(D) $\int_2^5 \sqrt{1 + (3x + \sin x)^2}dx$

Solution

The correct answer is (C)

$$f(x) = 3x + \sin x$$

$$f'(x) = 3 + \cos x$$

Since

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

We have

$$L = \int_a^b \sqrt{1 + (3 + \cos x)^2}\, dx$$

3. The length of the curve $y = \sqrt{1 - x^2}$ from $x = 0$ to $x = 1$ is most nearly equal to

(A) 0.7854

(B) 1.414

(C) 1.518

(D) 1.571

Solution

The correct answer is (D)

One does not need to solve this problem using the length of arc formula. The curve $y = \sqrt{1 - x^2}$ is that of a circle of radius 1. From 0 to 1, it is the quarter length of the circumference of the circle.

$$L = \frac{2\pi r}{4}$$

$$= \frac{2\pi(1)}{4}$$

$$= \frac{\pi}{2}$$

$$= 1.571$$

Alternatively, using the arc length formula, we get

$$y = \sqrt{1 - x^2}$$

$$= (1 - x^2)^{\frac{1}{2}}$$

$$y' = \frac{1}{2}(1 - x^2)^{-\frac{1}{2}}(-2x)$$

$$= -\frac{x}{\sqrt{1 - x^2}}$$

$$L = \int_a^b \sqrt{1 + (y')^2}\, dx$$

$$= \int_0^1 \sqrt{1 + \left(-\frac{x}{\sqrt{1 - x^2}}\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + \frac{x^2}{1 - x^2}}\, dx$$

$$= \int_0^1 \sqrt{\frac{1 - x^2 \ + x^2}{1 - x^2}}\, dx$$

$$= \int_0^1 \sqrt{\frac{1}{1 - x^2}}\, dx$$

$$= \left[\sin^{-1} x\right]_0^1$$

$$= \sin^{-1} 1 - \sin^{-1} 0$$

$$= \frac{\pi}{2} - 0$$

$$= 1.570796327$$

$$\approx 1.571$$

4. A robotic drawing pencil is using linear spline interpolation to trace a path consecutively through three data points (4,20), (6,10) and (9,25). What is the length of the path that the pencil traces if it begins at the first data point and ends at the last?

(A) 15.297
(B) 10.198
(C) 7.071
(D) 25.495

Solution

The correct answer is (D)

Using linear splines, we then have a piecewise continuous function made of two straight lines making the path. The path follows a straight line from $(4,20)$ to $(6,10)$ and another goes from $(6,10)$ to $(9,25)$. The length of a straight line, l between two points (x_1, y_1) and (x_2, y_2) is given by

$$l = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

Hence, the total length of the two straight lines combined is

$$L = \sqrt{(10 - 20)^2 + (6 - 4)^2} + \sqrt{(25 - 10)^2 + (9 - 6)^2}$$

$$= 10.198 + 15.297$$

$$= 25.495$$

5. A path is traversed consecutively through three points (2,4), (5,11), (8,3) using a quadratic polynomial. Estimate the exact length of the path up to at least 4 significant digits. Use only MATLAB or some other program to do this problem - do not do it manually as it will take time.

(A) 16.701

(B) 6.0827

(C) 16.160

(D) 51.387

Solution

The correct answer is (A)

Using MATLAB, we find that the quadratic polynomial that goes through $(2,4), (5,11)$, and $(8,3)$ is

$$y = -0.8333x^2 + 8.1667x - 9$$

Then

$$y' = -1.666x + 8.1667$$

$$L = \int_a^b \sqrt{1 + (y')^2}\, dx$$

$$= \int_2^8 \sqrt{1 + (-1.666x + 8.1667)^2}\, dx$$

$$= 16.701$$

6. A cable hangs between two poles that are 50 feet apart. The shape of the cable is given by

$$f(x) = a\cosh\left(\frac{x}{a}\right), -25 \le x \le 25,$$

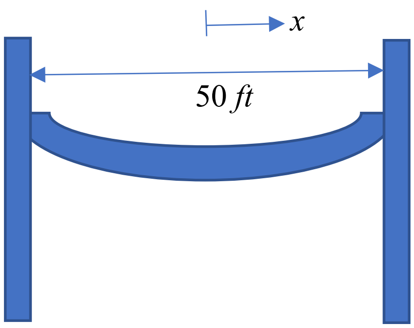
where a is dependent on the tension in the cable and the weight per unit length of the cable, and x is measured between the poles but from the center of cable. If the length of the cable is 60 feet, the formula that will allow us to find the value of a is

(A) $2\,a^2\sinh\left(\frac{25}{a}\right) = 60$

(B) $2\,a\sinh\left(\frac{25}{a}\right) = 60$

(C) $2\,a\sinh\left(\frac{25}{a}\right) = 50$

(D) $2\,a^2\sinh\left(\frac{25}{a}\right) = 50$



Solution

The correct answer is (B)

$$f(x) = a\cosh\left(\frac{x}{a}\right)$$

$$f'(x) = a\sinh\left(\frac{x}{a}\right)\left(\frac{1}{a}\right)$$

$$= \sinh\left(\frac{x}{a}\right)$$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$$

$$= \int_{-25}^{25} \sqrt{1 + \sinh^2\left(\frac{x}{a}\right)}\,dx$$

$$= \int_{-25}^{25} \sqrt{\cosh^2\left(\frac{x}{a}\right)}\,dx$$

$$= \int_{-25}^{25} \cosh\left(\frac{x}{a}\right)dx$$

$$= \left[a \sinh\left(\frac{x}{a}\right) \right]_{-25}^{25}$$

$$= a \sinh\left(\frac{25}{a}\right) - a \sinh\left(-\frac{25}{a}\right)$$

$$= a \sinh\left(\frac{25}{a}\right) + a \sinh\left(\frac{25}{a}\right)$$

$$= 2a \sinh\left(\frac{25}{a}\right)$$

Since

$$L = 60$$

we get the nonlinear equation as

$$60 = 2a \sinh\left(\frac{25}{a}\right)$$