Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
Bisection Method
Nonlinear Equations
COMPLETE SOLUTION SET

1. The bisection method of finding roots of nonlinear equations falls under the category of a (an) _________ method.
   (A) open
   (B) bracketing
   (C) random
   (D) graphical

Solution
The correct answer is (B).

The bisection method is a bracketing method since it is based on finding the root between two guesses that bracket the root, that is, where the real continuous function $f(x)$ in the equation $f(x)=0$ changes sign between the two guesses.

2. If for a real continuous function $f(x)$, $f(a)f(b)<0$, then in the range of $[a,b]$ for $f(x)=0$, there is (are)

(A) one root

(B) an undeterminable number of roots

(C) no root

(D) at least one root

Solution

The correct answer is (D).

If $f(a)f(b)<0$, then $f(a)$ and $f(b)$ have opposite signs. Since $f(x)$ is continuous between a and b, the function needs to cross the x-axis. The point where the function $f(x)$ crosses the x-axis is the root of the equation $f(x)=0$.

3. Assuming an initial bracket of $[1,5]$, the second (at the end of 2 iterations) iterative value of the root of $te^{-t} - 0.3 = 0$ using the bisection method is

(A) 0
(B) 1.5
(C) 2
(D) 3

Solution
The correct answer is (C).

$$f(t) = te^{-t} - 0.3$$

If the initial bracket is [1,5] then

$$t_u = 5$$
$$t_\ell = 1$$

Check to see if the function changes sign between t_ℓ and t_u

$$f(t_u) = 5e^{-5} - 0.3$$
$$= -0.2663$$
$$f(t_\ell) = 1e^{-1} - 0.3$$
$$= 0.0679$$

Hence,

$$f(t_u)f(t_\ell) = f(5)f(1)$$
$$= (-0.2663)(0.0679)$$
$$= -0.0181 < 0$$

So there is at least one root between t_ℓ and t_u.

Iteration 1

The estimate of the root is

$$t_m = \frac{t_\ell + t_u}{2}$$
$$= \frac{1+5}{2}$$
$$= 3$$
$$f(t_m) = 3e^{-3} - 0.3$$
$$= -0.1506$$

Thus,

$$f(t_\ell)f(t_m) = f(1)f(3)$$
$$= (0.0679)(-0.1506)$$
$$= -0.0102 < 0$$

The root lies between t_ℓ and t_m, so the new upper and lower guesses for the root are

$$t_\ell = t_\ell = 1$$

$$t_u = t_m = 3$$

<u>Iteration 2</u>

The estimate of the root is

$$\begin{aligned} t_m &= \frac{t_\ell + t_u}{2} \\ &= \frac{1+3}{2} \\ &= 2 \end{aligned}$$

4. To find the root of $f(x) = 0$, a scientist is using the bisection method. At the beginning of an iteration, the lower and upper guesses of the root are x_l and x_u. At the end of the iteration, the absolute relative approximate error in the estimated value of the root would be

(A) $\left|\dfrac{x_u}{x_u + x_\ell}\right|$

(B) $\left|\dfrac{x_\ell}{x_u + x_\ell}\right|$

(C) $\left|\dfrac{x_u - x_\ell}{x_u + x_\ell}\right|$

(D) $\left|\dfrac{x_u + x_\ell}{x_u - x_\ell}\right|$

Solution

The correct answer is (C).

The absolute relative approximate error is

$$|\in_a| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right|$$

where

$$x_m^{new} = \frac{x_\ell + x_u}{2}$$

If

$$x_m^{old} = x_\ell$$

$$|\in_a| = \left|\frac{\dfrac{x_\ell + x_u}{2} - x_\ell}{\dfrac{x_\ell + x_u}{2}}\right|$$

$$= \left|\frac{(x_\ell + x_u) - 2x_\ell}{x_\ell + x_u}\right|$$

$$= \left|\frac{x_u - x_\ell}{x_u + x_\ell}\right|$$

If

$$x_m^{old} = x_u$$

$$\left|\in_a\right| = \left|\frac{\dfrac{x_\ell + x_u}{2} - x_u}{\dfrac{x_\ell + x_u}{2}}\right|$$

$$= \left|\frac{(x_\ell + x_u) - 2x_u}{x_\ell + x_u}\right|$$

$$= \left|\frac{x_l - x_u}{x_u + x_\ell}\right|$$

The answer is the same whether $x_m^{old} = x_\ell \ or \ x_u$ as x_m is exactly in the middle of x_ℓ and x_u.

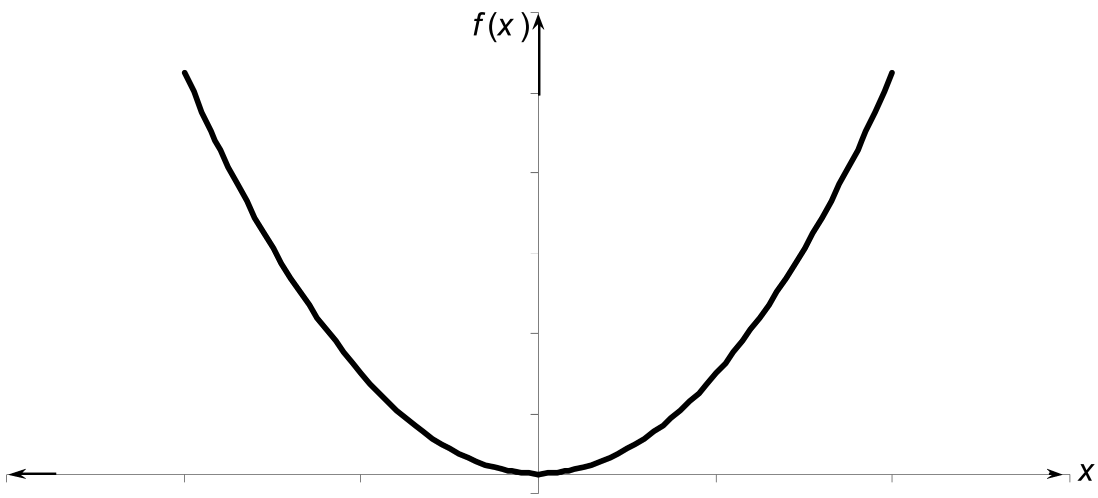
5. For an equation like $x^2 = 0$, a root exists at $x = 0$. The bisection method cannot be adopted to solve this equation in spite of the root existing at $x = 0$ because the function $f(x) = x^2$

(A) is a polynomial
(B) has repeated roots at $x = 0$
(C) is always non-negative
(D) has a slope equal to zero at $x = 0$

Solution

The correct answer is (C).

Since $f(x) = x^2$ will never be negative, the statement $f(x_u)f(x_\ell) < 0$ will never be true. Therefore, no interval $[x_\ell, x_u]$ will contain the root of $x^2 = 0$.



6. The ideal gas law is given by

$$pv = RT$$

where p is the pressure, v is the specific volume, R is the universal gas constant, and T is the absolute temperature. This equation is only accurate for a limited range of pressure and temperature. Vander Waals came up with an equation that was accurate for larger ranges of pressure and temperature given by

$$\left(p + \frac{a}{v^2}\right)(v - b) = RT$$

where a and b are empirical constants dependent on a particular gas. Given the value of $R = 0.08$, $a = 3.592$, $b = 0.04267$, $p = 10$ and $T = 300$ (assume all units are consistent), one is going to find the specific volume, v, for the above values. Without finding the solution from the Vander Waals equation, what would be a good initial guess for v?

(A) 0
(B) 1.2
(C) 2.4
(D) 3.6

Solution

The correct answer is (C).

From the physics of the problem, the initial guess can be found from the linear relationship of the ideal gas law

$$pv = RT$$

$$v = \frac{RT}{p}$$

$$= \frac{0.08 \times 300}{10}$$

$$= 2.4$$