Holistic Numerical Methods Institute
committed to bringing numerical methods to undergraduates

Multiple-Choice Test
Differentiation of Continuous Functions
Differentiation
COMPLETE SOLUTION SET

1. The definition of the first derivative of a function $f(x)$ is

(A) $f'(x) = \dfrac{f(x+\Delta x)+f(x)}{\Delta x}$

(B) $f'(x) = \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$

(C) $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x)+f(x)}{\Delta x}$

(D) $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$

Solution

The correct answer is (D).

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Choice (B) is incorrect as it is an approximate method to calculate the first derivative of a function $f(x)$. In fact, choice (B) is the forward divided difference method of approximately calculating the first derivative of a function.

2. The exact derivative of $f(x) = x^3$ at $x = 5$ is most nearly

(A) 25.00
(B) 75.00
(C) 106.25
(D) 125.00

Solution

The correct answer is (B).

$$
\begin{aligned}
f(x) &= x^3 \\
f'(x) &= 3x^2 \\
f'(5) &= 3(5)^2 \\
&= 75.00
\end{aligned}
$$

3. Using the forwarded divided difference approximation with a step size of 0.2, the derivative of $f(x) = 5e^{2.3x}$ at $x = 1.25$ is

(A) 163.4
(B) 203.8
(C) 211.1
(D) 258.8

Solution
The correct answer is (D).

The forward divided difference approximation is

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

where

$$x = 1.25$$
$$\Delta x = 0.2$$

Thus,

$$\begin{aligned} f'(1.25) &\approx \frac{f(1.25+0.2) - f(1.25)}{0.2} \\ &= \frac{f(1.45) - f(1.25)}{0.2} \\ &= \frac{5e^{2.3(1.45)} - 5e^{2.3(1.25)}}{0.2} \\ &= 258.8 \end{aligned}$$

4. A student finds the numerical value of $\frac{d}{dx}(e^x) = 20.220$ at $x = 3$ using a step size of 0.2. Which of the following methods did the student use to conduct the differentiation?

(A) Backward divided difference
(B) Calculus, that is, exact
(C) Central divided difference
(D) Forward divided difference

Solution

The correct answer is (C).

Choice (A)

The backward divided difference approximation is

$$f'(x) \approx \frac{f(x) - f(x - \Delta x)}{\Delta x}$$

where

$$x = 3$$
$$\Delta x = 0.2$$

Thus,

$$\begin{aligned} f'(3) &\approx \frac{f(3) - f(3 - 0.2)}{(0.2)} \\ &= \frac{f(3) - f(2.8)}{(0.2)} \\ &= \frac{e^3 - e^{2.8}}{0.2} \\ &= 18.204 \end{aligned}$$

Choice (B)

Using calculus,

$$\frac{d}{dx}(e^x) = e^x$$

Thus,

$$\begin{aligned} f'(3) &= e^3 \\ &= 20.086 \end{aligned}$$

Choice (C)

The central divided difference approximation is

$$f'(x) \approx \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

where

$$x = 3$$
$$\Delta x = 0.2$$

Thus,

$$f'(3) \approx \frac{f(3+0.2)-f(3-0.2)}{2(0.2)}$$
$$= \frac{f(3.2)-f(2.8)}{2(0.2)}$$
$$= \frac{e^{3.2}-e^{2.8}}{0.4}$$
$$= 20.220$$

Choice (D)

The forward divided difference approximation is

$$f'(x) \approx \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

where

$$x = 3$$
$$\Delta x = 0.2$$

Thus,

$$f'(3) \approx \frac{f(3+0.2)-f(3)}{(0.2)}$$
$$= \frac{f(3.2)-f(3)}{(0.2)}$$
$$= \frac{e^{3.2}-e^{3}}{0.2}$$
$$= 22.235$$

5. Using the backward divided difference approximation, $\frac{d}{dx}(e^x) = 4.3715$ at $x = 1.5$ for a step size of 0.05. If you keep halving the step size to find $\frac{d}{dx}(e^x)$ at $x = 1.5$ before two significant digits can be considered to be at least correct in your answer, the step size would be (you cannot use the exact value to determine the answer)

(A) 0.05/2
(B) 0.05/4
(C) 0.05/8
(D) 0.05/16

Solution

The correct answer is (C).

The equation for the backward difference approximation is

$$f'(x) \approx \frac{f(x_i) - f(x_i - \Delta x)}{\Delta x}$$

Half the step size and find the value of

$$\frac{d}{dx}(e^x) \text{ at } x = 1.5$$

$$\Delta x = 0.05/2$$
$$= 0.025$$

$$f'(1.5) = \frac{f(1.5) - f(1.475)}{0.025}$$
$$= \frac{e^{1.5} - e^{1.475}}{0.025}$$
$$= 4.4261$$

The absolute relative approximate error is

$$|\in_a| = \left|\frac{4.4261 - 4.3715}{4.4261}\right| \times 100$$
$$= 1.2345\%$$

Since $1.2345\% \le 0.5 \times 10^{2-m}$ for a maximum integer value of $m = 1$, there is at least one significant digit correct. But, we are looking for 2 significant digits so we must halve the previous step size and find the backward difference approximation again.

$$\Delta x = 0.05/4$$
$$= 0.0125$$

$$f'(1.5) = \frac{f(1.5) - f(1.4875)}{0.0125}$$
$$= \frac{e^{1.5} - e^{1.4875}}{0.0125}$$

$$= 4.4538$$

The absolute relative approximate error is

$$|\in_a| = \left|\frac{4.4538 - 4.4261}{4.4538}\right| \times 100$$

$$= 0.62111\%$$

Since for $0.62111\% \leq 0.5 \times 10^{2-m}$ for a maximum integer value of $m = 1$, again, there is only at least one significant digit correct. We must halve the previous step size and find the backward difference again.

$$\Delta x = 0.05/8$$

$$= 0.00625$$

$$f'(1.5) = \frac{f(1.5) - f(1.49375)}{0.00625}$$

$$= \frac{e^{1.5} - e^{1.49375}}{0.00625}$$

$$= 4.4677$$

The absolute relative approximate error is

$$|\in_a| = \left|\frac{4.4677 - 4.4538}{4.4677}\right| \times 100$$

$$= 0.31153\%$$

Since $0.31153\% \leq 0.5 \times 10^{2-m}$ for a maximum integer value of $m = 2$. Now, there are at least two significant digits correct in the iteration. Thus, the answer is

$$\Delta x = 0.05/8$$

6. The heat transfer rate q over a surface is given by

$$q = -kA\frac{dT}{dy}$$

where

k = thermal conductivity $\left(\frac{\text{J}}{\text{s}\cdot\text{m}\cdot\text{K}}\right)$

A = surface area $\left(\text{m}^2\right)$

T = temperature (K)

y = distance normal to the surface (m)

Given

$$k = 0.025\frac{\text{J}}{\text{s}\cdot\text{m}\cdot\text{K}}$$

$$A = 3\,\text{m}^2$$

the temperature T over the surface varies as

$$T = -1493y^3 + 2200y^2 - 1076y + 500$$

The heat transfer rate q at the surface most nearly is

(A) -1076 W
(B) 37.5 W
(C) 80.7 W
(D) 500 W

Solution

The correct answer is (C).

$$\frac{dT}{dy} = \frac{d}{dy}(-1493y^3) + \frac{d}{dy}(2200y^2) - \frac{d}{dy}(1076y) + \frac{d}{dy}(500)$$

$$= -4479y^2 + 4400y - 1076$$

Thus,

$$q = -0.025\left(\frac{\text{J}}{\text{s}\cdot\text{m}\cdot\text{K}}\right)\times 3\left(\text{m}^2\right)\times(-4479y^2 + 4400y - 1076)\left(\frac{\text{K}}{\text{m}}\right)$$

Since $y = 0\,\text{m}$ at the surface,

$$q = -0.025\left(\frac{\text{J}}{\text{s}\cdot\text{m}\cdot\text{K}}\right)\times 3\left(\text{m}^2\right)\times(-4479(0)^2 + 4400(0) - 1076)\left(\frac{\text{K}}{\text{m}}\right)$$

$$= -0.025\left(\frac{\text{J}}{\text{s}\cdot\text{m}\cdot\text{K}}\right)\times 3\left(\text{m}^2\right)\times(-1076)\left(\frac{\text{K}}{\text{m}}\right)$$

$$= 80.7\,\frac{\text{J}}{\text{s}} \text{ or W}$$