**Problem Set#1**
**Multiple Choice Test**
**Chapter 01.07 Taylors Series Revisited**
**COMPLETE SOLUTION SET**

1. The coefficient of the $x^5$ term in the Maclaurin polynomial for $\sin(2x)$ is
   (A) 0
   (B) 0.0083333
   (C) 0.016667
   (D) 0.26667

**Solution**
*The correct answer is (D).*

The Maclaurin series for $\sin(2x)$ is

$$\begin{aligned}\sin(2x) &= 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \cdots \\ &= 2x - \frac{8x^3}{6} + \frac{32x^5}{120} + \cdots \\ &= 2x - 1.3333x^3 + 0.26667x^5 + \cdots\end{aligned}$$

Hence, the coefficient of the $x^5$ term is 0.26667.

2. Given $f(3)=6$, $f'(3)=8$, $f''(3)=11$, and all other higher order derivatives of $f(x)$ are zero at $x=3$, and assuming the function and all its derivatives exist and are continuous between $x=3$ and $x=7$, the value of $f(7)$ is

(A) 38.000
(B) 79.500
(C) 126.00
(D) 331.50

**Solution**

*The correct answer is (C).*

The Taylor series is given by

$$f(x+h)=f(x)+f'(x)h+\frac{f''(x)}{2!}h^2+\frac{f'''(x)}{3!}h^3+\cdots$$

$$x=3,\ h=7-3=4$$

$$f(3+4)=f(3)+f'(3)4+\frac{f''(3)}{2!}4^2+\frac{f'''(3)}{3!}4^3+\cdots$$

$$f(7)=f(3)+f'(3)4+\frac{f''(3)}{2!}4^2+\frac{f'''(3)}{3!}4^3+\ldots$$

Since all the derivatives higher than second are zero,

$$f(7)=f(3)+f'(3)4+\frac{f''(3)}{2!}4^2$$

$$=6+8\times4+\frac{11}{2!}4^2$$

$$=126$$

3. Given that $y(x)$ is the solution to $\frac{dy}{dx} = y^3 + 2$, $y(0) = 3$ the value of $y(0.2)$ from a second order Taylor polynomial around $x$=0 is

(A) 4.400
(B) 8.800
(C) 24.46
(D) 29.00

**Solution**

*The correct answer is (C).*

The second order Taylor polynomial is

$$y(x+h) = y(x) + y'(x)h + \frac{y''(x)}{2!}h^2$$

$$x = 0, \ h = 0.2 - 0 = 0.2$$

$$y(0+0.2) = y(0) + y'(0) \times 0.2 + \frac{y''(0)}{2!}0.2^2$$

$$y(0.2) = y(0) + y'(0) \times 0.2 + y''(0) \times 0.02$$

$$y(0) = 3$$

$$y'(x) = y^3 + 2$$

$$\begin{aligned} y'(0) &= 3^3 + 2 \\ &= 29 \end{aligned}$$

$$\begin{aligned} y''(x) &= 3y^2 \frac{dy}{dx} \\ &= 3y^2\left(y^3 + 2\right) \end{aligned}$$

$$\begin{aligned} y''(0) &= 3(3)^2\left(3^3 + 2\right) \\ &= 783 \end{aligned}$$

$$\begin{aligned} y(0.2) &= 3 + 29 \times 0.2 + 783 \times 0.02 \\ &= 24.46 \end{aligned}$$

4. The series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}4^n$ is a Maclaurin series for the following function

(A) $\cos(x)$
(B) $\cos(2x)$
(C) $\sin(x)$
(D) $\sin(2x)$

**Solution**

*The correct answer is (B).*

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$

$$\cos(2x) = \sum_{n=0}^{\infty}(-1)^n \frac{(2x)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{2^{2n}x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{4^n x^{2n}}{(2n)!}$$

5. The function $erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ is called the error function. It is used in the field of probability and cannot be calculated exactly. However, one can expand the integrand as a Taylor polynomial and conduct integration. The approximate value of $erf(2.0)$ using the first three terms of the Taylor series around $t=0$ is

(A) -0.75225
(B) 0.99532
(C) 1.5330
(D) 2.8586

**Solution**

*The correct answer is (A).*

Rewrite the integral as

$$erf(x)=\int_0^x \frac{2}{\sqrt{\pi}}e^{-t^2}dt$$

The first three terms of the Taylor series for $f(t)=\frac{2}{\sqrt{\pi}}e^{-t^2}$ around $t=0$ are

$$f(t)=\frac{2}{\sqrt{\pi}}e^{-t^2}$$

$$f(0)=\frac{2}{\sqrt{\pi}}e^{-0^2}$$

$$=\frac{2}{\sqrt{\pi}}$$

$$f'(t)=\frac{2}{\sqrt{\pi}}e^{-t^2}(-2t)$$

$$f'(0)=\frac{2}{\sqrt{\pi}}e^{-t^2}(-2(0))$$

$$=0$$

$$f''(t)=\frac{2}{\sqrt{\pi}}e^{-t^2}(-2t)(-2t)+\frac{2}{\sqrt{\pi}}e^{-t^2}(-2)$$

$$f''(0)=\frac{2}{\sqrt{\pi}}e^{-0^2}(-2(0))(-2(0))+\frac{2}{\sqrt{\pi}}e^{-0^2}(-2)$$

$$=-\frac{4}{\sqrt{\pi}}$$

The first three terms of the Taylor series are

$$f(x+h)=f(x)+f'(x)h+\frac{f''(x)}{2!}h^2$$

$$f(0+h) = f(0) + f'(0)h + \frac{f''(0)}{2!}h^2$$

$$f(h) = f(0) + f'(0)h + \frac{f''(0)}{2!}h^2$$

$$= \frac{2}{\sqrt{\pi}} + 0(h) - \frac{4}{\sqrt{\pi}}\frac{h^2}{2!}$$

$$= \frac{2}{\sqrt{\pi}} - \frac{4}{\sqrt{\pi}}\frac{h^2}{2!}$$

$$= \frac{2}{\sqrt{\pi}} - \frac{2}{\sqrt{\pi}}h^2$$

$$\frac{2}{\sqrt{\pi}}e^{-h^2} \approx \frac{2}{\sqrt{\pi}} - \frac{2}{\sqrt{\pi}}h^2 \text{, or}$$

$$\frac{2}{\sqrt{\pi}}e^{-x^2} \approx \frac{2}{\sqrt{\pi}} - \frac{2}{\sqrt{\pi}}x^2$$

Hence

$$erf(x) \approx \int_0^x \left( \frac{2}{\sqrt{\pi}} - \frac{2}{\sqrt{\pi}}t^2 \right) dt$$

$$= \left[ \frac{2}{\sqrt{\pi}}t - \frac{2}{\sqrt{\pi}}\frac{t^3}{3} \right]_0^x$$

$$= \frac{2}{\sqrt{\pi}}x - \frac{2}{\sqrt{\pi}}\frac{x^3}{3}$$

$$erf(2) = \frac{2}{\sqrt{\pi}}(2) - \frac{2}{\sqrt{\pi}}\frac{2^3}{3}$$

$$= -0.75225$$

Note: Compare with the exact value of $erf(2)$

6. Using the remainder of Maclaurin polynomial of $n^{th}$ order for $f(x)$ defined as

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c), \quad n \geq 0,\ 0 \leq c \leq x$$

the order of the Maclaurin polynomial at least required to get an absolute true error of at most $10^{-6}$ in the calculation of $\sin(0.1)$ is (do not use the exact value of $\sin(0.1)$ or $\cos(0.1)$ to find the answer, but the knowledge that $|\sin(x)| \leq 1$ and $|\cos(x)| \leq 1$).

(A) 3
(B) 5
(C) 7
(D) 9

**Solution**

*The correct answer is (B).*

$$R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c),\ n \geq 0,\ 0 \leq c \leq x$$

$$R_n(0.1) = \frac{(0.1)^{n+1}}{(n+1)!} f^{(n+1)}(c),\ n \geq 0,\ 0 \leq c \leq 0.1$$

Since derivatives of $f(x)$ are simply $\sin(x)$ and $\cos(x)$, and

$|\sin(x)| \leq 1$ and $|\cos(x)| \leq 1$

$$\left|f^{(n+1)}(c)\right| \leq 1$$

$$R_n(0.1) \leq \frac{(0.1)^{n+1}}{(n+1)!}(1)$$

$$= \frac{(0.1)^{n+1}}{(n+1)!}$$

So when is

$$R_n(0.1) < 10^{-6}$$

$$\frac{(0.1)^{n+1}}{(n+1)!} < 10^{-6}$$

$$n \geq 4$$

But since the Maclaurin series for $\sin(x)$ only includes odd terms, $n \geq 5$.