Chapter 08.04
Runge-Kutta 4th Order Method for Ordinary Differential Equations

After reading this chapter, you should be able to
1. develop Runge-Kutta 4th order method for solving ordinary differential equations,
2. find the effect size of step size has on the solution,
3. know the formulas for other versions of the Runge-Kutta 4th order method

What is the Runge-Kutta 4th order method?
Runge-Kutta 4th order method is a numerical technique used to solve ordinary differential equation of the form
\[
\frac{dy}{dx} = f(x, y), y(0) = y_0
\]
So only first order ordinary differential equations can be solved by using the Runge-Kutta 4th order method. In other sections, we have discussed how Euler and Runge-Kutta methods are used to solve higher order ordinary differential equations or coupled (simultaneous) differential equations.

How does one write a first order differential equation in the above form?

Example 1
Rewrite
\[
\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5
\]
in
\[
\frac{dy}{dx} = f(x, y), \quad y(0) = y_0 \text{ form.}
\]
Solution
\[
\frac{dy}{dx} + 2y = 1.3e^{-x}, \quad y(0) = 5
\]
\[
\frac{dy}{dx} = 1.3e^{-x} - 2y, \quad y(0) = 5
\]
In this case
\[f(x, y) = 1.3e^{-x} - 2y\]

Example 2
Rewrite
\[e^y \frac{dy}{dx} + x^2 y^2 = 2 \sin(3x), \quad y(0) = 5\]
in
\[\frac{dy}{dx} = f(x, y), \quad y(0) = y_0 \] form.

Solution
\[e^y \frac{dy}{dx} + x^2 y^2 = 2 \sin(3x), \quad y(0) = 5\]
\[
\frac{dy}{dx} = \frac{2 \sin(3x) - x^2 y^2}{e^y}, \quad y(0) = 5
\]
In this case
\[f(x, y) = \frac{2 \sin(3x) - x^2 y^2}{e^y}\]
The Runge-Kutta 4th order method is based on the following
\[y_{i+1} = y_i + (a_1k_1 + a_2k_2 + a_3k_3 + a_4k_4)h\]
where knowing the value of \( y = y_i \) at \( x_i \), we can find the value of \( y = y_{i+1} \) at \( x_{i+1} \), and
\[h = x_{i+1} - x_i\]
Equation (1) is equated to the first five terms of Taylor series
\[y_{i+1} = y_i + \frac{dy}{dx}\bigg|_{x_i,y_i} (x_{i+1} - x_i) + \frac{1}{2!} \frac{d^2y}{dx^2}\bigg|_{x_i,y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!} \frac{d^3y}{dx^3}\bigg|_{x_i,y_i} (x_{i+1} - x_i)^3 + \frac{1}{4!} \frac{d^4y}{dx^4}\bigg|_{x_i,y_i} (x_{i+1} - x_i)^4\]
(2)
Knowing that \( \frac{dy}{dx} = f(x, y) \) and \( x_{i+1} - x_i = h\)
\[y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!} f'(x_i, y_i)h^2 + \frac{1}{3!} f''(x_i, y_i)h^3 + \frac{1}{4!} f'''(x_i, y_i)h^4\]
(3)
Based on equating Equation (2) and Equation (3), one of the popular solutions used is
\[y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h\]
(4)
Runge-Kutta 4th Order Method

\[ k_1 = f(x_i, y_i) \]  \hspace{2cm} (5a)

\[ k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h\right) \]  \hspace{2cm} (5b)

\[ k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2h\right) \]  \hspace{2cm} (5c)

\[ k_4 = f\left(x_i + h, y_i + k_3h\right) \]  \hspace{2cm} (5d)

Example 3

The open loop response, that is, the speed of the motor to a voltage input of 20V, assuming a system without damping is

\[ 20 = (0.02) \frac{dw}{dt} + (0.06)w. \]

If the initial speed is zero \((w(0) = 0)\), and using the Runge-Kutta 4th order method, what is the speed at \(t = 0.8s\)? Assume a step size of \(h = 0.4s\).

Solution

\[ \frac{dw}{dt} = 1000 - 3w \]

\[ f(t, w) = 1000 - 3w \]

\[ w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \]

For \(i = 0, t_0 = 0, w_0 = 0\)

\[ k_1 = f(t_0, w_0) \]
\[ = f(0,0) \]
\[ = 1000 - 3 \times 0 \]
\[ = 1000 \]

\[ k_2 = f\left(t_0 + \frac{1}{2}h, w_0 + \frac{1}{2}k_1h\right) \]
\[ = f\left(0 + \left(\frac{1}{2} \times 0.4\right), 0 + \left(\frac{1}{2} (1000) \times 0.4\right)\right) \]
\[ = f(0.2, 200) \]
\[ = 1000 - 3 \times 200 \]
\[ = 400 \]

\[ k_3 = f\left(t_0 + \frac{1}{2}h, w_0 + \frac{1}{2}k_2h\right) \]
\[ = f\left(0 + \left(\frac{1}{2} \times 0.4\right), 0 + \left(\frac{1}{2} (400) \times 0.4\right)\right) \]
\[ = f(0.2, 80) \]
\[ = 1000 - 3 \times 80 \]
\[ k_4 = f(t_0 + h, w_0 + k_3 h) \]
\[ = f(0 + (0.4), 0 + ((760) \times 0.4)) \]
\[ = f(0.4, 304) \]
\[ = 1000 - 3 \times 304 \]
\[ = 88 \]

\[ w_i = w_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) h \]
\[ = 0 + \frac{1}{6} \left( 1000 + 2 \times 400 + 2 \times 760 + (88) \right) \times 0.4 \]
\[ = 0 + \frac{1}{6} (3408) \times 0.4 \]
\[ = 227.2 \text{ rad/s} \]

\( w_i \) is the approximate speed of the motor at

\[ t = t_1 = t_0 + h = 0 + 0.4 = 0.4 \text{ s} \]

\( w(0.4) \approx w_1 = 227.2 \text{ rad/s} \)

For \( i = 1, t_1 = 0.4, w_1 = 227.2 \)

\[ k_1 = f(t_1, w_1) \]
\[ = f(0.4, 227.2) \]
\[ = 1000 - 3 \times 227.2 \]
\[ = 318.4 \]

\[ k_2 = f\left(t_1 + \frac{1}{2} h, w_1 + \frac{1}{2} k_1 h\right) \]
\[ = f\left(0.4 + \frac{1}{2} (0.4), 227.2 + \frac{1}{2} (318.4) \times 0.4\right) \]
\[ = f(0.6, 290.88) \]
\[ = 1000 - 3 \times 290.88 \]
\[ = 127.36 \]

\[ k_3 = f\left(t_1 + \frac{1}{2} h, w_1 + \frac{1}{2} k_2 h\right) \]
\[ = f\left(0.4 + \frac{1}{2} (0.4), 227.2 + \frac{1}{2} (127.36) \times 0.4\right) \]
\[ = f(0.6, 252.67) \]
\[ = 1000 - 3 \times 252.67 \]
\[ = 241.98 \]

\[ k_4 = f(t_1 + h, w_1 + k_3 h) \]
\[ = f(0.4 + 0.4, 227.2 + (241.98 \times 0.4)) \]
\[ = f(0.8, 323.99) \]
\[ = 1000 - 3 \times 323.99 \]
\[ = 28.019 \]
\[ w_2 = w_i + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)h \]
\[ = 227.2 + \frac{1}{6} (318.4 + 2 \times (127.36) + 2 \times (241.98) + 28.019) \times 0.4 \]
\[ = 227.2 + \frac{1}{6} (1085.1) \times 0.4 \]
\[ = 299.54 \text{ rad/s} \]

\( w_2 \) is the approximate speed of the motor at
\[ t = t_2 = t_1 + h = 0.4 + 0.4 = 0.8 \text{ s} \]
\( w(0.8) \approx w_2 = 299.54 \text{ rad/s} \)

The exact solution of the ordinary differential equation is given by
\[ w(t) = \left( \frac{1000}{3} \right) - \left( \frac{1000}{3} \right) e^{-3t} \]

The solution to this nonlinear equation at \( t = 0.8 \text{ s} \) is
\[ w(0.8) = 303.09 \text{ rad/s} \]

Figure 1 compares the exact solution with the numerical solution using the Runge-Kutta 4th order method using different step sizes.

\[ \text{Figure 1} \text{ Comparison of Runge-Kutta 4th order method with exact solution for different step sizes.} \]
Table 1 and Figure 2 show the effect of step size on the value of the calculated speed of the motor at $t = 0.8\,\text{s}$.

**Table 1** Values of speed of the motor at 0.8 seconds for different step sizes.

| Step size, $h$ | $w(0.8)$ | $E_x$ | $|\epsilon|\%$ |
|---------------|-----------|-------|----------------|
| 0.8           | 147.20    | 155.89| 51.434         |
| 0.4           | 299.54    | 3.5535| 1.1724         |
| 0.2           | 302.96    | 0.12988| 0.042852      |
| 0.1           | 303.09    | 0.0062962| 0.0020773   |
| 0.05          | 303.09    | 0.00034702| 0.00011449  |

**Figure 2** Effect of step size in Runge-Kutta 4th order method.

In Figure 3, we are comparing the exact results with Euler’s method (Runge-Kutta 1st order method), Heun’s method (Runge-Kutta 2nd order method) and the Runge-Kutta 4th order method.
Figure 3 Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.