Direct Method of Interpolation

Industrial Engineering Majors

Authors: Autar Kaw, Jai Paul

http://numericalmethods.eng.usf.edu
Transforming Numerical Methods Education for STEM Undergraduates
Direct Method of Interpolation

http://numericalmethods.eng.usf.edu
What is Interpolation?

Given \((x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\), find the value of ‘\(y\)’ at a value of ‘\(x\)’ that is not given.

**Figure 1** Interpolation of discrete.
Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate
Direct Method

Given ‘n+1’ data points \((x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\), pass a polynomial of order ‘n’ through the data as given below:

\[
y = a_0 + a_1x + \ldots + a_nx^n.
\]

where \(a_0, a_1, \ldots, a_n\) are real constants.

- Set up ‘n+1’ equations to find ‘n+1’ constants.
- To find the value ‘y’ at a given value of ‘x’, simply substitute the value of ‘x’ in the above polynomial.
Example

A curve needs to be fit through the given points to fabricate the cam. If the cam follows a straight line profile between \( x = 1.28 \) to \( x = 0.66 \), what is the value of \( y \) at \( x=1.1 \)? Find using the direct method and linear interpolation.

<table>
<thead>
<tr>
<th>Point</th>
<th>( x ) (in.)</th>
<th>( y ) (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.20</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>1.28</td>
<td>0.88</td>
</tr>
<tr>
<td>3</td>
<td>0.66</td>
<td>1.14</td>
</tr>
<tr>
<td>4</td>
<td>0.00</td>
<td>1.20</td>
</tr>
<tr>
<td>5</td>
<td>−0.60</td>
<td>1.04</td>
</tr>
<tr>
<td>6</td>
<td>−1.04</td>
<td>0.60</td>
</tr>
<tr>
<td>7</td>
<td>−1.20</td>
<td>0.00</td>
</tr>
</tbody>
</table>
Linear Interpolation

\[ y(x) = a_0 + a_1 x \]
\[ y(1.28) = a_0 + a_1 (1.28) = 0.88 \]
\[ y(0.66) = a_0 + a_1 (0.66) = 1.14 \]

Solving the above two equations gives,

\[ a_0 = 1.4168 \quad a_1 = -0.41935 \]

Hence

\[ y(x) = 1.4168 - 0.41935x, \quad 0.66 \leq x \leq 1.28. \]

\[ y(1.10) = 1.4168 - 0.41935(1.10) = 0.95548 \text{ in.} \]
Example

A curve needs to be fit through the given points to fabricate the cam. If the cam follows a straight line profile between x = 1.28 to x = 0.66, what is the value of y at x=1.1 ? Find using the direct method and quadratic interpolation.

<table>
<thead>
<tr>
<th>Point</th>
<th>x (in.)</th>
<th>y (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.20</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>1.28</td>
<td>0.88</td>
</tr>
<tr>
<td>3</td>
<td>0.66</td>
<td>1.14</td>
</tr>
<tr>
<td>4</td>
<td>0.00</td>
<td>1.20</td>
</tr>
<tr>
<td>5</td>
<td>-0.60</td>
<td>1.04</td>
</tr>
<tr>
<td>6</td>
<td>-1.04</td>
<td>0.60</td>
</tr>
<tr>
<td>7</td>
<td>-1.20</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Cam Profile
Quadratic Interpolation

\[ y(x) = a_0 + a_1 x + a_2 x^2 \]

\[ y(2.20) = a_0 + a_1 (2.20) + a_2 (2.20)^2 = 0 \]

\[ y(1.28) = a_0 + a_1 (1.28) + a_2 (1.28)^2 = 0.88 \]

\[ y(0.66) = a_0 + a_1 (0.66) + a_2 (0.66)^2 = 1.14 \]

Solving the above three equations gives

\[ a_0 = 1.1221 \quad a_1 = 0.25734 \quad a_2 = -0.34881 \]
Quadratic Interpolation (contd)

\[ y(x) = 1.1221 + 0.25734x - 0.34881x^2, \quad 0.66 \leq x \leq 2.20 \]

\[ y(1.10) = 1.1221 + 0.25734(1.10) - 0.34881(1.10)^2 \]

\[ = 0.98311 \text{ in.} \]

The absolute relative approximate error obtained between the results from the first and second order polynomial is

\[
|\varepsilon_a| = \left| \frac{0.98311 - 0.95548}{0.98311} \right| \times 100
\]

\[ = 2.8100\% \]
## Comparison Table

<table>
<thead>
<tr>
<th>Order of Polynomial</th>
<th>Value of $y$ at $x = 1.10$</th>
<th>Absolute Relative Approximate Error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.98311</td>
<td>2.8100 %</td>
</tr>
<tr>
<td></td>
<td>0.95548</td>
<td></td>
</tr>
</tbody>
</table>
Example

A curve needs to be fit through the given points to fabricate the cam. If the cam follows a straight line profile between \( x = 1.28 \) to \( x = 0.66 \), what is the value of \( y \) at \( x = 1.1 \)? Find using the direct method and a sixth order polynomial.

<table>
<thead>
<tr>
<th>Point</th>
<th>( x ) (in.)</th>
<th>( y ) (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.20</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>1.28</td>
<td>0.88</td>
</tr>
<tr>
<td>3</td>
<td>0.66</td>
<td>1.14</td>
</tr>
<tr>
<td>4</td>
<td>0.00</td>
<td>1.20</td>
</tr>
<tr>
<td>5</td>
<td>-0.60</td>
<td>1.04</td>
</tr>
<tr>
<td>6</td>
<td>-1.04</td>
<td>0.60</td>
</tr>
<tr>
<td>7</td>
<td>-1.20</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Cam Profile

http://numericalmethods.eng.usf.edu
Sixth Order Interpolation

\[ y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \]

\[ y(2.20) = 0.00 = a_0 + a_1 (2.20) + a_2 (2.20)^2 + a_3 (2.20)^3 + a_4 (2.20)^4 + a_5 (2.20)^5 + a_6 (2.20)^6 \]

\[ y(1.28) = 0.88 = a_0 + a_1 (1.28) + a_2 (1.28)^2 + a_3 (1.28)^3 + a_4 (1.28)^4 + a_5 (1.28)^5 + a_6 (1.28)^6 \]

\[ y(0.66) = 1.14 = a_0 + a_1 (0.66) + a_2 (0.66)^2 + a_3 (0.66)^3 + a_4 (0.66)^4 + a_5 (0.66)^5 + a_6 (0.66)^6 \]

\[ y(0.00) = 1.20 = a_0 + a_1 (0.00) + a_2 (0.00)^2 + a_3 (0.00)^3 + a_4 (0.00)^4 + a_5 (0.00)^5 + a_6 (0.00)^6 \]

\[ y(-0.60) = 1.04 = a_0 + a_1 (-0.60) + a_2 (-0.60)^2 + a_3 (-0.60)^3 + a_4 (-0.60)^4 + a_5 (-0.60)^5 + a_6 (-0.60)^6 \]

\[ y(-1.04) = 0.60 = a_0 + a_1 (-1.04) + a_2 (-1.04)^2 + a_3 (-1.04)^3 + a_4 (-1.04)^4 + a_5 (-1.04)^5 + a_6 (-1.04)^6 \]

\[ y(-1.20) = 0.00 = a_0 + a_1 (-1.20) + a_2 (-1.20)^2 + a_3 (-1.20)^3 + a_4 (-1.20)^4 + a_5 (-1.20)^5 + a_6 (-1.20)^6 \]
Sixth Order Interpolation (contd)

Writing the seven equations in matrix form, we have

\[
\begin{bmatrix}
1 & 2.20 & 2.20^2 & 2.20^3 & 2.20^4 & 2.20^5 & 2.20^6 \\
1 & 1.28 & 1.28^2 & 1.28^3 & 1.28^4 & 1.28^5 & 1.28^6 \\
1 & 0.66 & 0.66^2 & 0.66^3 & 0.66^4 & 0.66^5 & 0.66^6 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -0.60 & 0.60^2 & -0.60^3 & 0.60^4 & -0.60^5 & 0.60^6 \\
1 & -1.04 & 1.04^2 & -1.04^3 & 1.04^4 & -1.04^5 & 1.04^6 \\
1 & -1.20 & 1.20^2 & -1.20^3 & 1.20^4 & -1.20^5 & 1.20^6 \\
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5 \\
a_6 \\
\end{bmatrix} =
\begin{bmatrix}
0.00 \\
0.88 \\
1.14 \\
1.20 \\
1.04 \\
0.60 \\
0.00 \\
\end{bmatrix}
\]
Sixth Order Polynomial (contd)

Solving the above seven equations gives

\[ a_0 = 1.2 \quad a_2 = -0.27255 \quad a_4 = 0.072013 \quad a_6 = -0.17103 \]
\[ a_1 = 0.25112 \quad a_3 = -0.56765 \quad a_5 = 0.45241 \]

\[ y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \]

\[ = 1.2 + 0.25112x - 0.27255x^2 - 0.56765x^3 \]
\[ + 0.072013x^4 + 0.45241x^5 - 0.17103x^6, \quad -1.20 \leq x \leq 2.20 \]
Sixth Order Polynomial (contd)

\[ y(x) = 1.2 + 0.25112x - 0.27255x^2 - 0.56765x^3 \\
+ 0.072013x^4 + 0.45241x^5 - 0.17103x^6, \quad -1.20 \leq x \leq 2.20 \]
Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/direct_method.html
THE END

http://numericalmethods.eng.usf.edu