Newton’s Divided Difference Polynomial Method of Interpolation

Computer Engineering Majors

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Transforming Numerical Methods Education for STEM Undergraduates
Newton’s Divided Difference Method of Interpolation

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What is Interpolation?

Given \((x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\), find the value of ‘y’ at a value of ‘x’ that is not given.
Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.
Newton’s Divided Difference Method

**Linear interpolation:** Given \((x_0, y_0), (x_1, y_1)\), pass a linear interpolant through the data

\[ f_1(x) = b_0 + b_1(x - x_0) \]

where

\[ b_0 = f(x_0) \]
\[ b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \]
Example

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a rectangular plate. The hole centers in the plate that describe the path the arm needs to take are given below.

If the laser is traversing from \( x = 2 \) to \( x = 4.25 \) in a linear path, find the value of \( y \) at \( x = 4 \) using the Newton’s Divided Difference method for linear interpolation.

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Figure 2  Location of holes on the rectangular plate.
Linear Interpolation

\[ y(x) = b_0 + b_1 (x - x_0) \]

\[ x_0 = 2.00, \; y(x_0) = 7.2 \]

\[ x_1 = 4.25, \; y(x_1) = 7.1 \]

\[ b_0 = y(x_0) = 7.2 \]

\[ b_1 = \frac{y(x_1) - y(x_0)}{x_1 - x_0} = \frac{7.1 - 7.2}{4.25 - 2.00} = -0.044444 \]

\[ y_s \]
\[ f(\text{range}) \]
\[ f(x_{\text{desired}}) \]

\[ x_0 - 10 \]
\[ x_s, \text{ range, } x_{\text{desired}} \]
\[ x_1 + 10 \]
Linear Interpolation (contd)

\[ y(x) = b_0 + b_1 (x - x_0) \]

\[ = 7.2 - 0.044444(x - 2.00), \quad 2.00 \leq x \leq 4.25 \]

At \( x = 4 \)

\[ x(4.00) = 7.2 - 0.044444(4.00 - 2.00) \]

\[ = 7.1111 \text{ in.} \]
Quadratic Interpolation

Given \((x_0, y_0), (x_1, y_1),\) and \((x_2, y_2),\) fit a quadratic interpolant through the data.

\[
f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)
\]

\[
b_0 = f(x_0)
\]

\[
b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}
\]

\[
b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}
\]
Example

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a rectangular plate. The hole centers in the plate that describe the path the arm needs to take are given below.

If the laser is traversing from \( x = 2 \) to \( x = 4.25 \) in a linear path, find the value of \( y \) at \( x = 4 \) using the Newton’s Divided Difference method for quadratic interpolation.

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Figure 2  Location of holes on the rectangular plate.
Quadratic Interpolation (contd)

\[ y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \]

\[ x_0 = 2.00, \quad y(x_0) = 7.2 \]

\[ x_1 = 4.25, \quad y(x_1) = 7.1 \]

\[ x_2 = 5.25, \quad y(x_2) = 6.0 \]
Quadratic Interpolation (contd)

\[ b_0 = y(x_0) \]

\[ = 7.2 \]

\[ b_1 = \frac{y(x_1) - y(x_0)}{x_1 - x_0} = \frac{7.1 - 7.2}{4.25 - 2.00} \]

\[ = -0.044444 \]

\[ y(x_2) - y(x_1) \quad y(x_1) - y(x_0) \]

\[ b_2 = \frac{x_2 - x_1}{x_2 - x_0} - \frac{x_1 - x_0}{x_2 - x_0} = \frac{6.0 - 7.1}{5.25 - 4.25} - \frac{7.1 - 7.2}{4.25 - 2.00} \]

\[ = \frac{-1.1 + 0.044444}{3.25} \]

\[ = -0.32479 \]
Quadratic Interpolation (contd)

\[ y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \]
\[ = 7.2 - 0.044444(x - 2.00) - 0.32479(x - 2.00)(x - 4.25), \quad 2.00 \leq x \leq 5.25 \]

At \( x = 4 \),

\[ y(4.00) = 7.2 - 0.044444(4.00 - 2.00) - 0.32479(4.00 - 2.00)(4.00 - 4.25) \]
\[ = 7.2735 \text{ in.} \]

The absolute relative approximate error \( \epsilon_a \) obtained between the results from the first and second order polynomial is

\[ |\epsilon_a| = \left| \frac{7.2735 - 7.1111}{7.2735} \right| \times 100 \]
\[ = 2.2327\% \]
General Form

\[ f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \]

where

\[ b_0 = f[x_0] = f(x_0) \]

\[ b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \]

\[ b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_2 - x_0} \]

Rewriting

\[ f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) \]
General Form

Given \((n + 1)\) data points, \((x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)\) as

\[ f_n(x) = b_0 + b_1(x - x_0) + \ldots + b_n(x - x_0)(x - x_1)\ldots(x - x_{n-1}) \]

where

\[ b_0 = f[x_0] \]
\[ b_1 = f[x_1, x_0] \]
\[ b_2 = f[x_2, x_1, x_0] \]
\[ \vdots \]
\[ b_{n-1} = f[x_{n-1}, x_{n-2}, \ldots, x_0] \]
\[ b_n = f[x_n, x_{n-1}, \ldots, x_0] \]
The third order polynomial, given \((x_0, y_0), (x_1, y_1), (x_2, y_2),\) and \((x_3, y_3),\) is

\[
f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)
+ f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)
\]
Example

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a rectangular plate. The hole centers in the plate that describe the path the arm needs to take are given below.

If the laser is traversing from $x = 2$ to $x = 4.25$ in a linear path, find the value of $y$ at $x = 4$ using the Newton’s Divided Difference method for a fifth order polynomial.

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Figure 2 Location of holes on the rectangular plate.
Example

The value of y profile is chosen as
\[
y(x) = b_0 + b_1 (x - x_0) + b_2 (x - x_0)(x - x_1) + b_3 (x - x_0)(x - x_1)(x - x_2) \\
+ b_4 (x - x_0)(x - x_1)(x - x_2)(x - x_3) + b_5 (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)
\]

Using the six points,
\[
\begin{align*}
  x_0 &= 2.00, & y(x_0) &= 7.2 \\
  x_1 &= 4.25, & y(x_1) &= 7.1 \\
  x_2 &= 5.25, & y(x_2) &= 6.0 \\
  x_3 &= 7.81, & y(x_3) &= 5.0 \\
  x_4 &= 9.20, & y(x_4) &= 3.5 \\
  x_5 &= 10.60, & y(x_5) &= 5.0
\end{align*}
\]

The values of the constants are found to be
\[
\begin{align*}
b_0 &= 7.2 & b_1 &= -0.044444 & b_2 &= -0.32479 \\
b_3 &= 0.090198 & b_4 &= -0.023009 & b_5 &= 0.0072923
\end{align*}
\]
Example

Hence

\[ y(x) = b_0 + b_1 (x - x_0) + b_2 (x - x_0)(x - x_1) + b_3 (x - x_0)(x - x_1)(x - x_2) + b_4 (x - x_0)(x - x_1)(x - x_2)(x - x_3) + b_5 (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4) \]

\[ = 7.2 - 0.04444(x - 2) - 0.32479(x - 2)(x - 4.25) + 0.090198(x - 2)(x - 4.25)(x - 5.25) - 0.023009(x - 2)(x - 4.25)(x - 5.25)(x - 7.81) + 0.0072923(x - 2)(x - 4.25)(x - 5.25)(x - 7.81)(x - 9.2) \]

\[ y(x) = -30.898 + 41.344x - 15.855x^2 + 2.7862x^3 - 0.23091x^4 + 0.0072923x^5, \quad 2 \leq x \leq 10.6 \]
Example

\[ y(x) = -30.898 + 41.344x - 15.855x^2 + 2.7862x^3 - 0.23091x^4 + 0.0072922x^5, \]

\[ 2 \leq x \leq 10.6 \]
Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/newton_divided_difference_method.html
THE END

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