Chapter 03.00D

Physical Problem for Nonlinear Equations
Computer Engineering

Problem Statement
Many super computers do not have a unit to divide numbers. But why? Well, a divide operation in modern computers can take 20 to 25 clock cycles, and that is five times what it takes for multiplication [1]. Instead, a divide unit, based on numerically solving a nonlinear equation, is developed. This allows for a faster divide operation. This is how it works.

If you want to find the value of $\frac{b}{a}$ where $b$ and $a$ are real numbers, one can look at

$$\frac{b}{a} = b \times \frac{1}{a} = b \times c$$

where

$$c = \frac{1}{a}$$

So, if one is able to find $c$, we only need to multiply $b$ and $c$ to find $\frac{b}{a}$. So how do we find $c$ without the divide unit.

Equation

$$c = \frac{1}{a}$$

can be written as an equation

$$f(c) = ac - 1 = 0$$

If one is able to find the root of this equation without using a division, then we have the value of $\frac{1}{a}$.

Although we do not explain the numerical methods of solving nonlinear equations in this section of the notes, it becomes necessary to do so in this example. The Newton-Raphson method of solving a nonlinear equation is used in finding $\frac{1}{a}$. The Newton Raphson method of solving an equation $f(c) = 0$ is given by the iterative formula

$$c_{i+1} = c_i - \frac{f(c_i)}{f'(c_i)},$$

where
$c_{i+1}$ is the new approximation of the root of $f(c) = 0$ and $c_i$ is the previous approximation of the root of $f(c) = 0$.

**What is the appropriate function to use to find the inverse of $a$?**

a) Using $f(c) = ac - 1 = 0$ gives $f'(c) = a$ and the Newton-Raphson method formula gives

$$
c_{i+1} = c_i - \frac{ac_i - 1}{a}
$$

$$
c_{i+1} = \frac{1}{a}
$$

This is of no use as it involves division.

b) Using $f(c) = a - \frac{1}{c} = 0$ gives $f'(c) = \frac{1}{c^2}$

$$
c_{i+1} = c_i - \frac{a - \frac{1}{c_i}}{\frac{1}{c_i^2}}
$$

$$
c_{i+1} = c_i - c_i^2 \left( a - \frac{1}{c_i} \right)
$$

$$
c_{i+1} = c_i \left( 2 - c_i a \right)
$$

This one is the acceptable iterative formula to find the inverse of $a$ as it does not involve division.

Starting with an initial guess for the inverse of $a$, one can find newer approximations by using the above iterative formula. Each iteration requires two multiplications and one subtraction. However, the number of iterations required to find the inverse of $a$ very much depends on the initial approximation. More accurate is the starting approximation, less number of iterations are required to find the inverse of $a$. Since the convergence of Newton Raphson method is quadratic, it may take up to six iterations to get an accurate reciprocal in double precision. By using look-up tables for the initial approximation, the number of iterations required can be reduced to two [2]. Also, the operation of $(2 - c_i a)$ may be carried in a fused multiply-subtract unit to further reduce the clock cycles needed for the computation.
Appendix A: Example of using Newton–Raphson method to find the inverse of a number.

Let us find \( \frac{1}{2.5} \)

The Newton–Raphson method formula is given by

\[
 c_{i+1} = c_i \left(2 - 2.5 c_i\right)
\]

Starting with estimate of inverse as \( c_0 = 0.5 \),

\[
 c_1 = 0.5 \left[2 - 2.5(0.5)\right] \\
 = 0.375
\]

\[
 c_2 = 0.375 \left[2 - 2.5(0.375)\right] \\
 = 0.3984
\]

\[
 c_3 = 0.3984 \left[2 - 2.5(0.3984)\right] \\
 = 0.3999
\]

\[
 c_4 = 0.3999 \left[2 - 2.5(0.3999)\right] \\
 = 0.4000
\]

It took four iterations to find the inverse of 2.5 correct up to 4 significant digits.