When solving a fixed-constant linear ordinary differential equation where the part of the homogeneous solution is same form as part of a possible particular solution, why do we get the next independent solution in the form of $x^n \times \textit{part of possible particular solution}$ **? Show this through an example.**

Let's suppose we want to solve the ordinary differential equation

$$2\frac{dy}{dx} + 4y = e^{-x}, \ y(0) = 7 \tag{1}$$

The characteristic equation is

$$2m + 4 = 0$$

The solution to the characteristic equation is

$$m = -2$$

and hence the homogenous part of the solution to the ordinary differential equation is

$$y_H = Ke^{-2x}$$

Since the forcing function is e^{-x}, the particular part of the solution is chosen as the form of the forcing function and its derivatives.

$$y_p = Ae^{-x}$$

Substitute this in the ordinary differential equation, we get

$$2\frac{dy_p}{dx} + 4y_p = e^{-x}$$

$$2\frac{d}{dx}(Ae^{-x}) + 4(Ae^{-x}) = e^{-x}$$

$$-2Ae^{-x} + 4Ae^{-x} = e^{-x}$$

$$2Ae^{-x} = e^{-x}$$

$$A = \frac{1}{2}$$

Hence

$$y_p = \frac{1}{2}e^{-x}$$

But what if the differential equation instead was

$$2\frac{dy}{dx} + 4y = e^{-2x}, \ y(0) = 7 \tag{2}$$

The only difference in equations (1) and (2) is that the forcing function is e^{-2x} instead of e^{-x}. The characteristic equation is

$$2m + 4 = 0$$

$$m = -2$$

giving the homogeneous solution as

$$y_H = Ke^{-2x}$$

Since the forcing function is e^{-2x}, the particular part of the solution chosen is

$$y_p = Ae^{-2x}$$

But this will not work in this case. Why? Substitute this in the ordinary differential equation, we get

$$2\frac{dy_p}{dx} + 4y_p = e^{-2x}$$

$$2\frac{d}{dx}(Ae^{-2x}) + 4(Ae^{-2x}) = e^{-2x}$$

$$-4Ae^{-2x} + 4Ae^{-2x} = e^{-2x}$$

$$0 = e^{-2x}$$

This is not possible. We should not have picked this form of solution as we know that Ke^{-2x} is part of the homogenous solution.

So what is the correct particular part of the solution to pick? We chose $y_p = Axe^{-2x}$

$$2\frac{dy_p}{dx} + 4y_p = e^{-2x}$$

$$2\frac{d}{dx}(Axe^{-2x}) + 4(Axe^{-2x}) = e^{-2x}$$

$$2(Ae^{-2x} - Axe^{-2x}) + 4Axe^{-2x} = e^{-2x}$$

$$2Ae^{-2x} - 4Axe^{-2x} + 4Axe^{-2x} = e^{-2x}$$

$$2Ae^{-2x} = e^{-2x}$$

$$A = \frac{1}{2}$$

Hence

$$y_p = \frac{1}{2} x e^{-2x}$$

But the question which students ask is that why do we multiply by *x*? <u>is that a random pick</u>. Why not multiply x^2 or sin(*x*) or something else?

See for the forcing function of equation (1)

$$2\frac{dy_p}{dx} + 4y_p = e^{-2x}$$

Multiply both sides by e^{2x}

$$2e^{2x}\frac{dy_p}{dx} + 4e^{2x}y_p = (e^{2x})e^{-2x}$$

$$\frac{d}{dx}(2e^{2x}y_p) = 1$$

$$2e^{2x}y_p = x$$

giving

$$y_p = \frac{1}{2} x e^{-2x}$$

and for the forcing function of equation (2)

$$2\frac{dy_p}{dx} + 4y_p = e^{-x}$$

Multiply both sides by e^{2x}

$$2e^{2x}\frac{dy_p}{dx} + 4e^{2x}y_p = (e^{2x})e^{-x}$$

$$\frac{d}{dx}(2e^{2x}y_p) = e^{x}$$

$$2e^{2x}y_p = e^{x}$$

giving

$$y_p = \frac{1}{2} e^{-x}$$