When solving a fixed-constant linear ordinary differential equation where the characteristic equation has repeated roots, why do we get the next independent solution in the form of $x^n e^{mx}$? Show this through an example.

Let's suppose we want to solve the ordinary differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

The characteristic equation is

$$m^2 + 6m^1 + 9m^0 = 0$$

$$m^2 + 6m + 9 = 0$$

$$(m+3)^2 = 0$$

The solution to the characteristic equation is

$$m = -3, -3$$

and the homogeneous part of the solution to the ordinary differential equation is

$$y_H = K_1 e^{-3x} + K_2 x e^{-3x}$$

Since the forcing function is zero, the particular part of the solution to the ordinary differential equation is

$$y_P = 0$$

The complete solution to the ordinary differential equation is

$$y = y_H + y_P$$

$$= K_1 e^{-3x} + K_2 x e^{-3x}$$

But why is $K_2 x e^{-3x}$ part of the solution? See the answer below.

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

can be rewritten as

$$\left(\frac{d}{dx} + 3\right)\left(\frac{d}{dx} + 3\right)y = 0 \qquad (1)$$

Let

$$\left(\frac{d}{dx} + 3\right)y = z \qquad (2)$$

Then equation (1) can be written as

$$\left(\frac{d}{dx}+3\right)z=0 \qquad (3)$$

Let

$$z=K_2e^{m_2x}$$

be a possible solution form of equation (3), then

$$\left(\frac{d}{dx}+3\right)\left(K_2e^{m_2x}\right)=0$$

$$K_2m_2e^{m_2x}+3K_2e^{m_2x}=0$$

$$K_2\left(m_2+3\right)e^{m_2x}=0$$

Then

$$m_2=-3$$

is a solution which is nontrivial.

So

$$z=K_2e^{-3x}$$

From equation (2)

$$\left(\frac{d}{dx}+3\right)y=z$$

$$\left(\frac{d}{dx}+3\right)y=K_2e^{-3x}$$

$$\frac{dy}{dx}+3y=K_2e^{-3x}$$

Multiply both sides by e^{3x}

$$e^{3x}\left(\frac{dy}{dx}+3y\right)=K_2e^{-3x}e^{3x}$$

$$e^{3x}\frac{dy}{dx}+3e^{3x}y=K_2$$

$$e^{3x}\frac{d}{dx}(y)+y\frac{d}{dx}(e^{3x})=K_2$$

By chain rule

$$\frac{d}{dx}\left(e^{3x}y\right)=K_2$$

Integrating both sides with respect to x, we get

$$e^{3x}y = K_2x + K_1$$

$$y = K_2xe^{-3x} + K_1e^{-3x}$$