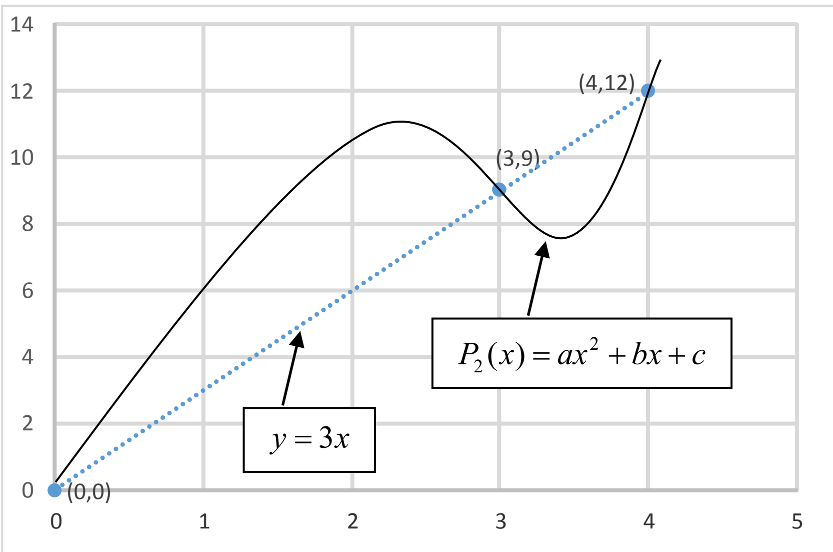
Problem

Through three data pairs (0,0), (3,9) and (4,12), an interpolating polynomial of order 2 or less is found to be $y=3x$. **Prove** that there is no other polynomial of order 2 or less that passes through these three points.

Solution

Let us use proof by contradiction. Let there be another polynomial other than $y=3x$ of order 2 or less that passes through the three data pairs (0,0), (3,9) and (4,12). Let this polynomial be

$$P_2(x) = ax^2 + bx + c\,.$$



Then polynomial

$$P_2(x) = ax^2 + bx + c$$

passes through (0,0), (3,9), (4,12). This gives us

$$a(0)^2 + b(0) + c = 0$$

$$a(3)^2 + b(3) + c = 9$$

$$a(4)^2 + b(4) + c = 12$$

In matrix form, the three equations can be written as

$$\begin{bmatrix} 0 & 0 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 12 \end{bmatrix}$$

By observation

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

is a solution to this set of 3 equations as

$$\begin{bmatrix} 0 & 0 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 12 \end{bmatrix}$$

But is this solution unique? The solution is unique if the coefficient matrix is nonsingular (also called invertible), that is,

$$\det\begin{bmatrix} 0 & 0 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{bmatrix} \neq 0$$

So, let us find the determinant of the coefficient matrix. We have

$$\det\begin{bmatrix} 0 & 0 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{bmatrix} = -\det\begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \quad \text{(after Exchanging column 3 with column 1)}$$

$$= -\det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 9 \\ 0 & 4 & 16 \end{bmatrix} \quad \text{(after Row2} - \text{Row1 and Row 3} - \text{Row1)}$$

$$= -\det\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 9 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{(after Row 3} - \frac{4}{3}\text{*Row 2)}$$

$$= -(1 \times 3 \times 4)$$
$$= -12 \neq 0$$

This goes to show that

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

is a unique solution to give

$$P_2(x) = ax^2 + bx + c$$
$$= 0x^2 + 3x + 0$$
$$= 3x$$

but this is also same as the given polynomial of y=3x. This shows that there is no other polynomial other than y=3x of order 2 or less that passes through these three points

Questions to Ponder

Can you apply forward elimination steps of Gaussian elimination to get the determinant of the coefficient matrix?

What theorems regarding determinants of matrices were applied to calculate the determinant of the coefficient matrix?