Problem Statement

The general form of Taylor series is given below.

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + f''''(x)\frac{h^4}{4!} + f'''''(x)\frac{h^5}{5!} + \ldots\ldots$$

Given $f(4)=0$ $f'(4)=7$ $f''(4)=10$ $f'''(4)=30$ and all other higher derivatives of $f(x)$ at $x=4$ are zero. The function $f(x)$ is of polynomial form,

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 .$$

Find the function $f(x)$.

Solution

Since

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

then

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2$$
$$f''(x) = 2a_2 + 6a_3 x$$
$$f'''(x) = 6a_3$$

Now using the given values of the function and its derivatives at x=4

$$f(4) = 0 = a_0 + a_1 4 + a_2 4^2 + a_3 4^3$$
$$f'(4) = 7 = a_1 + 2a_2 7 + 3a_3 7^2$$
$$f''(4) = 10 = 2a_2 + 6a_3 4$$
$$f'''(4) = 30 = 6a_3$$

gives

$$\begin{bmatrix} 1 & 4 & 16 & 64 \\ 0 & 1 & 14 & 147 \\ 0 & 0 & 2 & 24 \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 10 \\ 30 \end{bmatrix}$$

Using back substitution of the Gaussian Elimination Method algorithm (for more info, go to http://numericalmethods.eng.usf.edu/topics/gaussian_elimination.html), we get

$$a_3 = 5, \; a_2 = -55, \; a_1 = 207, \; a_0 = -268$$

Hence

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$
$$= -268 + 207x - 55x^2 + 5x^3$$

Questions

Confirm the given values of the function and its derivatives at x=4.

References

Taylor Series Revisited,
http://numericalmethods.eng.usf.edu/topics/taylor_series.html