**Problem Statement**

The general form of Taylor series is given below.

\[
f(x + h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + f''''(x)\frac{h^4}{4!} + f'''''(x)\frac{h^5}{5!} + \ldotsn
\]

Given \( f(4) = 0 \), \( f'(4) = 7 \), \( f''(4) = 10 \), \( f'''(4) = 30 \) and all other higher derivatives of \( f(x) \) at \( x = 4 \) are zero. The function \( f(x) \) is of polynomial form, \( f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \).

Find the function \( f(x) \).

**Solution**

Since

\[
f(x) = a_0 + a_1x + a_2x^2 + a_3x^3
\]

then

\[
\begin{align*}
f'(x) &= a_1 + 2a_2x + 3a_3x^2 \\
f''(x) &= 2a_2 + 6a_3x \\
f'''(x) &= 6a_3
\end{align*}
\]

Now using the given values of the function and its derivatives at \( x = 4 \)

\[
\begin{align*}
f(4) &= 0 = a_0 + a_14 + a_24^2 + a_34^3 \\
f'(4) &= 7 = a_1 + 2a_27 + 3a_37^2 \\
f''(4) &= 10 = 2a_2 + 6a_34 \\
f'''(4) &= 30 = 6a_3
\end{align*}
\]

gives

\[
\begin{bmatrix}
1 & 4 & 16 & 64 \\
0 & 1 & 14 & 147 \\
0 & 0 & 2 & 24 \\
0 & 0 & 0 & 6
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
7 \\
10 \\
30
\end{bmatrix}
\]

Using back substitution of the Gaussian Elimination Method algorithm (for more info, go to [http://numericalmethods.eng.usf.edu/topics/ gaussian_elimination.html](http://numericalmethods.eng.usf.edu/topics/ gaussian_elimination.html)), we get

\[
a_3 = 5, \quad a_2 = -55, \quad a_1 = 207, \quad a_0 = -268
\]

Hence

\[
f(x) = a_0 + a_1x + a_2x^2 + a_3x^3
= -268 + 207x - 55x^2 + 5x^3
\]

**Questions**

Confirm the given values of the function and its derivatives at \( x = 4 \).

**References**

Taylor Series Revisited, [http://numericalmethods.eng.usf.edu/topics/taylor_series.html](http://numericalmethods.eng.usf.edu/topics/taylor_series.html)