Solving First Order Linear ODE by Laplace Transforms

Example

Solve the following ordinary differential equation using Laplace Transforms

$$\frac{dy}{dx} + 0.4y = 3e^{-x} \ , \ y(0) = 5 \qquad (1)$$

Solution

Using

$$L\left(\frac{dy}{dx}\right) = sY(s) - y(0)$$

$$L(e^{-bx}) = \frac{1}{s+b}$$

then taking the Laplace transform of both sides, we get

$$L\left(\frac{dy}{dx} + 0.4y\right) = L\left(3e^{-x}\right)$$

$$[sY(s) - y(0)] + 0.4Y(s) = \frac{3}{s+1}$$

Using the initial condition, $y(0) = 5$ we get

$$[sY(s) - 5] + 0.4Y(s) = \frac{3}{s+1}$$

$$(s + 0.4)Y(s) = \frac{3}{s+1} + 5$$

$$(s + 0.4)Y(s) = \frac{5s+8}{s+1}$$

$$Y(s) = \frac{5s+8}{(s+1)(s+0.4)}$$

Writing the expression for $Y(s)$ in terms of partial fractions

$$\frac{5s+8}{(s+1)(s+0.4)} = \frac{A}{s+1} + \frac{B}{s+0.4}$$

$$\frac{5s+8}{(s+1)(s+0.4)} = \frac{As + 0.4A + Bs + B}{(s+1)(s+0.4)}$$

$$5s + 8 = As + 0.4A + Bs + B$$

Equating coefficients of s^1 and s^0 gives

$$A + B = 5$$

$$0.4A + B = 8$$

The solution to the above two simultaneous linear equations is

$$A = -5$$
$$B = 10$$

Then

$$Y(s) = \frac{-5}{s+1} + \frac{10}{s+0.4}$$

Taking the inverse Laplace transform on both sides

$$L^{-1}\{Y(s)\} = L^{-1}\left(\frac{-5}{s+1}\right) + L^{-1}\left(\frac{10}{s+0.4}\right)$$

Since

$$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

The solution is given by

$$y(x) = -5e^{-x} + 10e^{-0.4x}$$

The complete solution is

$$y = 10e^{-0.4x} - 5e^{-x}$$

At $x = 3$,

$$y(3) = 10e^{-0.4(3)} - 5e^{-3}$$
$$= 2.763$$