Clearing up the confusion about diagonally dominant matrices – Part 4

What is an irreducible diagonally dominant matrix?

A $n \times n$ square matrix $[A]$ is an irreducible diagonally dominant matrix if

[A] is irreducible,

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}| \text{ for } i = 1,2,.....,n \text{, and}$$

$$|a_{ii}| > \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}| \text{ for at least one row, } i = 1,2,.....,n$$

The second condition means that for each row, the absolute value (also called magnitude) of the diagonal element is greater than or equal to the sum of the absolute values of the rest of the elements of that row. The third condition means that for at least one row, the absolute value (also called magnitude) of the diagonal element is greater than the sum of the absolute values of the rest of the elements of that row.

Example 1

Give examples of matrices that are irreducibly diagonally dominant and those that are not irreducibly diagonally dominant.

Solution

The matrix

$$[A] = \begin{bmatrix} 15 & 6 & 7 \\ 2 & -4.1 & -2 \\ 3 & 2 & 6 \end{bmatrix}$$

is an irreducible diagonally dominant matrix.

Why? Because the answer to every question below is Yes.

Is [A] irreducible? Yes.

Row 1: Is $|a_{11}| \geq |a_{12}| + |a_{13}|$? Yes, because

$|a_{11}| = |15|, |a_{12}| + |a_{13}| = |6| + |7| = 13, 15 \geq 13.$

Row 2: Is $|a_{22}| \geq |a_{21}| + |a_{23}|$? Yes, because

$|a_{22}| = |-4.1| = 4.1, \ |a_{21}| + |a_{23}| = |2| + |-2| = 4, \ 4.1 \geq 4$

Row 3: Is $|a_{33}| \geq |a_{31}| + |a_{32}|$? Yes, because

$|a_{33}| = |6|, |a_{31}| + |a_{32}| = |3| + |2| = 5, \ 6 \geq 5$

Is the inequality satisfied strictly for at least one row? Yes, it is satisfied for Rows 1, 2 and 3.

The matrix

$$[A] = \begin{bmatrix} -15 & 6 & 9 \\ 2 & -4 & -2 \\ 3 & -2 & 5 \end{bmatrix}$$

is a not an irreducible diagonally dominant matrix.
Why? Because the answer to every question below is *not* a Yes.

Is [A] irreducible? Yes.
Row 1: Is $|a_{11}| \geq |a_{12}| + |a_{13}|$? Yes, because
$|a_{11}| = |15|, |a_{12}| + |a_{13}| = |6| + |9| = 15, 15 \geq 15.$
Row 2: Is $|a_{22}| \geq |a_{21}| + |a_{23}|$? Yes, because
$|a_{22}| = |-4| = 4,\ \ |a_{21}| + |a_{23}| = |2| + |-2| = 4,\ 4 \geq 4$
Row 3: Is $|a_{33}| \geq |a_{31}| + |a_{32}|$? Yes, because
$|a_{33}| = |5|, |a_{31}| + |a_{32}| = |3| + |2| = 5,\ 5 \geq 5$
Is the inequality satisfied strictly for at least one row? No.

The matrix

$$[A] = \begin{bmatrix} -15 & 6 & 9 \\ 2 & -4.1 & -2 \\ 3 & -2 & 5 \end{bmatrix}$$

is an irreducible diagonally dominant matrix.
Why? Because the answer to every question below is Yes.

Is [A] irreducible? Yes.
Row 1: Is $|a_{11}| \geq |a_{12}| + |a_{13}|$? Yes, because
$|a_{11}| = |15|, |a_{12}| + |a_{13}| = |6| + |9| = 15, 15 \geq 15.$
Row 2: Is $|a_{22}| \geq |a_{21}| + |a_{23}|$? Yes, because
$|a_{22}| = |-4.1| = 4,\ \ |a_{21}| + |a_{23}| = |2| + |-2| = 4,\ 4.1 \geq 4$
Row 3: Is $|a_{33}| \geq |a_{31}| + |a_{32}|$? Yes, because
$|a_{33}| = |5|, |a_{31}| + |a_{32}| = |3| + |2| = 5,\ 5 \geq 5$
Is the inequality satisfied strictly for at least one row? Yes, it is satisfied for Row 2.

The matrix

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

is *not* an irreducible diagonally dominant matrix.
Why? Because the answer to every question below is *not* a Yes.

Is [A] irreducible? Yes.

Row 1: Is $|a_{11}| \geq |a_{12}| + |a_{13}|$? Yes, because

$|a_{11}| = |25|,\ |a_{12}| + |a_{13}| = |5| + |1| = 6,\ 25 \geq 6$

Row 2: Is $|a_{22}| \geq |a_{21}| + |a_{23}|$? No, because

$|a_{22}| = |8| = 8,\ |a_{21}| + |a_{23}| = |64| + |1| = 65,\ 8 < 65$

Row 3: Is $|a_{33}| \geq |a_{31}| + |a_{32}|$? No, because

$|a_{33}| = |1|, |a_{31}| + |a_{32}| = |144| + |12| = 156,\ 1 < 156$

There is no need to check for strict inequality condition..