Clearing up the confusion about diagonally dominant matrices – Part 1

In my inexperienced youthful days and going by then available textbooks, I myself have been a propagator of creating confusion about definitions of diagonally dominant matrices.

The confusion stems from valid phrases such as "diagonally dominant matrices", "strictly diagonally dominant matrices", "weakly diagonal dominant matrices", and "irreducibly diagonally dominant matrices". Confusion then also occurs from the properties (such as real or complex nature of eigenvalues, nonsingularity of matrices, non-requirement of partial pivoting to conduct LU factorization, guaranteed convergence of certain iterative methods of solving simultaneous linear equations, etc) that are true when diagonal dominance of a certain type is occurring. I am hoping that I clear this up in a series of upcoming blogs with examples and mathematical implications.

If any of the experts disagree or can make this series of blogs more categorical or clearer, please contact me at kaw@usf.edu

What is a diagonally dominant matrix?

A $n \times n$ square matrix $[A]$ is a diagonally dominant matrix if

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ i \neq j}}^{n} |\, a_{ij} \,| \text{ for } i = 1,2,.....,n$$

that is, for each row, the absolute value (also called magnitude) of the diagonal element is greater than or equal to the sum of the absolute values of the rest of the elements of that row.

Example 1

Give examples of matrices that are diagonally dominant and those that are not diagonally dominant.

Solution

The matrix

$$[A] = \begin{bmatrix} 15 & 6 & 7 \\ 2 & -4.1 & -2 \\ 3 & 2 & 6 \end{bmatrix}$$

is a diagonally dominant matrix.

Why? Because for each and every row, the answer to the question below is Yes.

Row 1: Is $|a_{11}| \geq |a_{12}| + |a_{13}|$? Yes, because

$|a_{11}| = |15|, |a_{12}| + |a_{13}| = |6| + |7| = 13, 15 \geq 13.$

Row 2: Is $|a_{22}| \geq |a_{21}| + |a_{23}|$? Yes, because

$|a_{22}| = |-4.1| = 4.1,\ \ |a_{21}| + |a_{23}| = |2| + |-2| = 4,\ 4.1 \geq 4$

Row 3: Is $|a_{33}| \geq |a_{31}| + |a_{32}|$? Yes, because

$|a_{33}| = |6|, |a_{31}| + |a_{32}| = |3| + |2| = 5,\ 6 \geq 5$

The matrix

$$[A] = \begin{bmatrix} -15 & 6 & 9 \\ 2 & -4 & -2 \\ 3 & -2 & 5 \end{bmatrix}$$

is a diagonally dominant matrix.
Why? Because for each and every row, the answer to the question below is Yes.

Row 1: Is $|a_{11}| \geq |a_{12}| + |a_{13}|$? Yes, because

$|a_{11}| = |15|, |a_{12}| + |a_{13}| = |6| + |9| = 15, 15 \geq 15.$

Row 2: Is $|a_{22}| \geq |a_{21}| + |a_{23}|$? Yes, because

$|a_{22}| = |-4| = 4,\ \ |a_{21}| + |a_{23}| = |2| + |-2| = 4,\ 4 \geq 4$

Row 3: Is $|a_{33}| \geq |a_{31}| + |a_{32}|$? Yes, because

$|a_{33}| = |5|, |a_{31}| + |a_{32}| = |3| + |2| = 5,\ 5 \geq 5$

The matrix

$$[A] = \begin{bmatrix} -15 & 6 & 9 \\ 2 & -4.1 & -2 \\ 3 & -2 & 5 \end{bmatrix}$$

is a diagonally dominant matrix.
Why? Because for each and every row, the answer to the question below is Yes.

Row 1: Is $|a_{11}| \geq |a_{12}| + |a_{13}|$? Yes, because

$|a_{11}| = |15|, |a_{12}| + |a_{13}| = |6| + |9| = 15, 15 \geq 15.$

Row 2: Is $|a_{22}| \geq |a_{21}| + |a_{23}|$? Yes, because

$|a_{22}| = |-4.1| = 4,\ \ |a_{21}| + |a_{23}| = |2| + |-2| = 4,\ 4.1 \geq 4$

Row 3: Is $|a_{33}| \geq |a_{31}| + |a_{32}|$? Yes, because

$|a_{33}| = |5|, |a_{31}| + |a_{32}| = |3| + |2| = 5,\ 5 \geq 5$

The matrix

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

is *not* a diagonally dominant matrix.

Why? Because for each and every row, the answer to the question below is not a Yes.

Row 1: Is $|a_{11}| \geq |a_{12}| + |a_{13}|$? Yes, because

$|a_{11}| = |25|,\ |a_{12}| + |a_{13}| = |5| + |1| = 6,\ 25 \geq 6$

Row 2: Is $|a_{22}| \geq |a_{21}| + |a_{23}|$? No, because

$|a_{22}| = |8| = 8,\ |a_{21}| + |a_{23}| = |64| + |1| = 65,\ 8 < 65$

Row 3: Is $|a_{33}| \geq |a_{31}| + |a_{32}|$? No, because

$|a_{33}| = |1|, |a_{31}| + |a_{32}| = |144| + |12| = 156,\ 1 < 156$